PHD

Finite automata and composite realisations.

Kidson, D.

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FINITE AUTOMATA

AND

COMPOSITE REALISATIONS

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SUMMARY

The theory of finite automata provides a formal approach to the design of sequential circuits, assuming the sequential aspect of the realisation to be in the form of bistables. No formal approach has been developed, however, to take advantage of the various sequential units available in MSI (Medium Scale Integration) form. The problem can be viewed as that of "decomposing" the objective automaton into an interconnection of MSI sequential units, and this is the approach adopted in the present study.

However the study of such "composite realisations" raises fundamental problems, for example what does an objective automaton represent? Moreover, how is an objective automaton to be formulated? It is also essential to clarify what is meant by a "realisation" of an objective automaton, so that in forming a "composite realisation" the basic aim is clearly understood.

The initial aim in the present study, however, is to consider even more fundamental problems. It would seem that finite-automata theory can be developed from just a few essential concepts, furthermore these concepts are closely interrelated so a unified appreciation can be gained. By adopting this approach, the theory of finite automata can be developed in close association with more general abstract algebra, and can be developed with regard to axiomatic set theory and universal algebra.

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PREFACE

Finite-automata theory provides a formal basis for designing sequential circuits [Lewin; Miller], using a representation of the design objective in the form of a finite automaton. State-reduction procedures can then be used to seek a more efficient expression of the design objective, and finally this "objective" automaton is translated into hardware, by formalising an assignment of the state, input and output symbols to appropriate codes. Finite-automata theory is also of interest in considering the limitations of sequential circuits. The theory shows that the behaviour of sequential circuits is strictly limited, so that a design objective cannot always be expressed as a finite automaton and cannot always be translated into a sequential circuit. In addition automata theory relates sequential circuits to Turing machines, neural networks, artifical languages and other discrete-parameter systems, and the theory also has an attraction of its own, as an exercise in formal reasoning and applied abstract algebra.

If the theory is established using unrelated ideas, however, a disjointed appreciation results. The alternative approach is to consider the fundamental concepts on which a study of finite automata can be based, and to pay particular attention to the way these concepts interrelate. Clearly any formal study will require the rules of logical inference, furthermore an understanding of set theory will be necessary, since discrete parameters are involved. Once logic and set theory are accepted as essential, however, the study of finite automata can be developed using few additional concepts. These concepts then recur throughout the theory, so that separate aspects of the study are seen to be closely related. For example automaton reduction, automaton realisation and automaton decomposition seem to have little in common, and yet the "weak homomorphism" concept [Yoeli] is of crucial importance in each case.

The initial aim of the present study is to consider fundamental concepts on which the theory of finite automata can be based. It appears there are just five such fundamental concepts, however they are closely interrelated, and the relationships are just as important as the concepts considered separately. This provides a unified view of finite-automata theory, and it is intended to consider automaton reduction, state-compatibility, and other familiar concepts using this approach.

Having established this approach to the theory of finite automata, the problem of automaton realisation can be considered. The translation of automata into elegant hardware is given extensive treatment in state-assignment studies [Haring]. It is well known that the assignment of codes to the automaton state, input and output symbols dictates the interdependency of the circuit parameters, and that this influences both the complexity of the combinational circuitry [Lewin] and the form of the circuit

[Hartmanis; Stearns & Hartmanis]. Conventional state-assignment studies are not always of interest to the practicing engineer however, since the sequential aspect need not be implemented using bistables. Instead the designer might seek to use the various sequential units, such as counters and shift registers, available in Medium Scale Integration (MSI) form. Despite the availability of such units no formal design approach has been developed, and the designer is justified in using an informal approach from the outset.

Consequently, it is desired to present a formal approach to automaton realisation using MSI units. The approach is based on automaton decomposition [Hartmanis & Stearns; Yoeli], and aims to formalise a composite realisation of the objective automaton, in the form of interconnected MSI units. The study will be independent of any particular MSI family, instead the aim is to consider the general problem of automaton realisation using interconnected stock units. The cascade, direct-product, and more complex compositions will be considered, and it will also be necessary to clarify the "realisation" concept. In the case of a completely-specified objective, a "realisation" must have a submachine equivalent to the objective automaton. The realisation of a partially-specified objective, however, is less straightforward.

Considering now the separate chapter contents, the aim in the first chapter is to establish fundamental concepts on which the theory of finite automata can be based, and to consider the way these concepts are related. In the second chapter systems of state transitions are formalised as "semiautomata", where a semiautomaton is a unary algebra with a mapping over the state set associated with each input symbol. In considering semiautomata the concepts and results from universal algebra [Cohn; Gratzer] are directly applicable, consequently this representation is retained in formalising the "automaton" concept. This approach establishes links between automaton concepts and those associated with groups, rings and other algebras, and again the way the concepts relate is of particular interest. It is also important to introduce expressive symbology, especially that relating to sequences of inputs or "input tapes", and that relating to the associated state and output mappings.

The third chapter is devoted to the expression of design objectives in finite-automaton form. This aspect of the design procedure is generally neglected, and the aim is to consider an objective automaton as a "finite state" expression of an objective translation from input tapes to outputs. The problem of forming such an objective automaton, using the "event graphs" associated with regular expressions [Ott & Feinstein], will be considered, and it is intended to relate this method to the Nerode theorem [Nerode; Rabin & Scott].

The fourth chapter presents an appreciation of the realisation concept, and aims to develop a formal approach to the realisation of objective automata. In particular, the problem of assessing stock units as direct realisations of objective automata is considered. The fifth chapter introduces the cascade and direct-product constructions using partial algebras, and in each case the relationship between the composite system and the component algebras is considered. The final chapter is devoted to the realisation of objective automata in composite form, as interconnections of stock units. The aim is to give formal proofs associated with this approach, indeed the development of formal reasoning concerning automaton realisation is considered the real challenge.

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FINITE AUTOMATA

AND

COMPOSITE REALISATIONS

HOMOMORPHISM

(Greek, from "the same" & "form")

- A resemblance of form

[The Shorter Oxford English Dictionary]

CONTENTS

CHAPTER ONE: Introduction

Automata theory is often presented as a sequence of unrelated ideas, little attempt being made to distinguish crucially important concepts from those of passing interest. However certain ideas appear time and time again, and the study of these ideas, and the way they interrelate, gives a unified appreciation. The present aim is to introduce preserved covers, preserved relations, image systems, homomorphism, and weak homomorphism, these being perhaps the most important of the recurring themes.

To appreciate what is meant by a "preserved cover" consider figure 1.1, where the graph nodes represent the memory conditions or "states" of a discrete-parameter sequential system, and the arcs represent transitions from one state to another, in response to applied inputs.

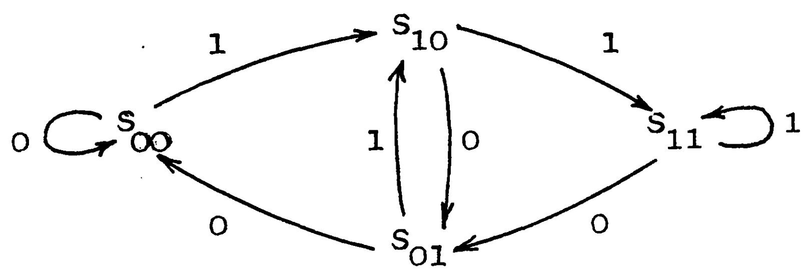


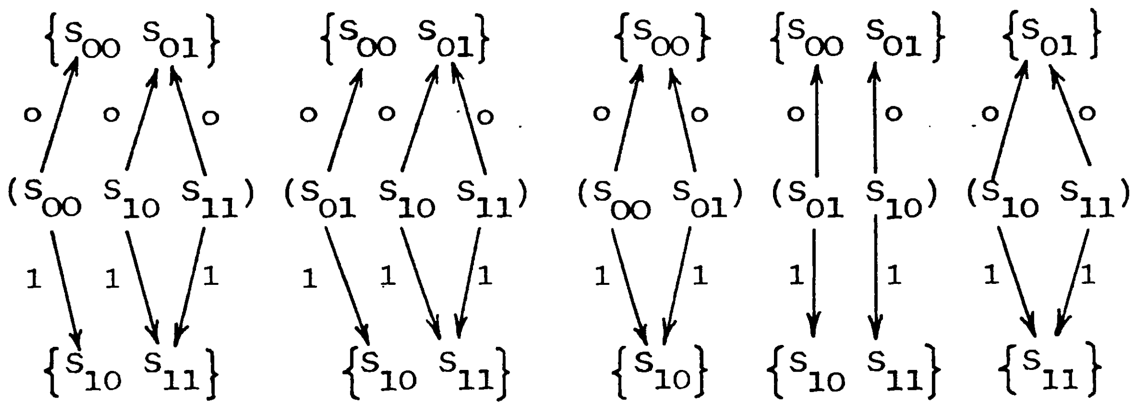
Figure 1.1 Transition system T

For example state S_{00} has state S_{10} as "1-successor", that is input 1 assigns state S_{00} to successor-state S_{10}, and this is represented as an arc labelled 1 leading from state S_{00} to state S_{10}. In fact the graph represents the state transitions of a two-stage shift register, with state S_{00} representing the state-variable code $\langle 00 \rangle$, and so on. For convenience this system of state transitions has been designated T, and the states form a set $S=\{S_{00}\ S_{01}\ S_{10}\ S_{11}\}$, the "state set" of the transition system.

The idea of a partition of a set such as $S=\{S_{00}\ S_{01}\ S_{10}\ S_{11}\}$ is encountered in set theory, for example $\{\{S_{00}\ S_{01}\}\ \{S_{10}\ S_{11}\}\}$ is a partition of set S, and in general a partition of a given set is a set of subsets of the given set, so that each element of the given set appears in one, and only one, of these subsets. Clearly the subsets forming a partition cannot have elements in common, that is the subsets must be "disjoint". For example $\{\{S_{00}\ S_{10}\ S_{11}\}\ \{S_{01}\ S_{10}\ S_{11}\}\}$ is not a partition of set $S=\{S_{00}\ S_{01}\ S_{10}\ S_{11}\}$, for although each element of S appears in some subset these subsets have elements in common. This is an example of a "cover" of the set S, and in general a cover of a given set is a set of subsets of the given set so that each element is contained in at least one of these subsets. The subsets $\{S_{00}\ S_{10}\ S_{11}\}$ and $\{S_{01}\ S_{10}\ S_{11}\}$ forming the above cover will be called the "blocks" of the cover, and for convenience the cover will be represented as $(S_{00}\ S_{10}\ S_{11})\ (S_{01}\ S_{10}\ S_{11})$.

Reconsidering the above transition system T, it has been observed that state S_{00} has state S_{10} as 1-successor, and the graph also shows S_{11} to be the 1-successor of state S_{10}, and S_{11} to be the 1-successor of state S_{11}.

Consequently the cover block $(S_{00}\ S_{10}\ S_{11})$ is converted to $\{S_{10}\ S_{11}\}$ for input 1, this being the set formed by the 1-successors of the states within this block. This subset $\{S_{10}\ S_{11}\}$ of the state set S will be called the "image" of block $(S_{00}\ S_{10}\ S_{11})$ for input 1, or more concisely the "1-image" of the block, and in general the 1-image of a given cover block is the set formed by the 1-successors of the states within the block, and similarly for the 0-image. In the case of the cover $(S_{00}\ S_{10}\ S_{11})$ $(S_{01}\ S_{10}\ S_{11})$ of the state set of transition system T, the image of each block for each input is evident from figure 1.2(a). Here the 0-successor of each state is indicated above the state, and the 1-successor is written below. Clearly the block $(S_{00}\ S_{10}\ S_{11})$ has 0-image $\{S_{00}\ S_{01}\}$ and has 1-image $\{S_{10}\ S_{11}\}$, similarly the 0-image of block $(S_{01}\ S_{10}\ S_{11})$ is $\{S_{00}\ S_{01}\}$ and the 1-image is $\{S_{10}\ S_{11}\}$.



(a) (b)

Figure 1.2

By a "preserved cover" of the transition system T is meant a cover of the state set S, so that the image of any block, for any input, is a subset of at least one block of the cover. Clearly cover $(S_{00}\ S_{10}\ S_{11})\ (S_{01}\ S_{10}\ S_{11})$ is not a preserved cover, since figure 1.2(a) shows that the O-image of block $(S_{00}\ S_{10}\ S_{11})$ is $\{S_{00}\ S_{01}\}$, and this is not a subset of block $(S_{00}\ S_{10}\ S_{11})$ nor of block $(S_{01}\ S_{10}\ S_{11})$. However the cover $\psi = (S_{00}\ S_{01})(S_{01}\ S_{10})\ (S_{10}\ S_{11})$ of state-set $S = \{S_{00}\ S_{01}\ S_{10}\ S_{11}\}$ is a preserved cover of transition system T, and this is evident from figure 1.2(b). The image of any block of this cover for any input is a subset of at least one cover block, for example the O-image of block $(S_{00}\ S_{01})$ is $\{S_{00}\}$ and this is a subset of block $(S_{00}\ S_{01})$, and the 1-image of block $(S_{00}\ S_{01})$ is $\{S_{10}\}$, where $\{S_{10}\}$ is a subset of block $(S_{01}\ S_{10})$ and of block $(S_{10}\ S_{11})$. This is just one of the many preserved covers of transition system T, and existing procedures [Booth] can be applied to determine all the preserved covers of any given transition system.

Any cover of a given set defines a relationship between the elements of the set, whereby two elements are considered related whenever one belongs to a block containing the other. Considering in particular the above cover $\psi = (S_{00}\ S_{01})(S_{01}\ S_{10})\ (S_{10}\ S_{11})$ of the set S, in this cover S_{00} is related to S_{01} since S_{00} belongs to the block $(S_{00}\ S_{01})$ containing S_{01}. This relationship can be represented as $S_{00}\ R_{\psi}\ S_{01}$, so that $S_{00}\ R_{\psi}\ S_{01}$ expresses that S_{00} belongs to a block of cover ψ containing S_{01}, and in the same symbology $S_{01}\ R_{\psi}\ S_{00}$ since S_{01} belongs to a block containing S_{00}. Clearly the relation R_{ψ} is symmetric, that is $s_i\ R_{\psi}\ s_j$ always implies $s_j\ R_{\psi}\ s_i$, and the relation R_{ψ} is also reflexive, that is $s_i\ R_{\psi}\ s_i$ for any state s_i from the state-set S. For convenience the relation R_{ψ} can be represented as a graph, as shown in figure 1.3, where the arc from S_{00} to S_{01} represents $S_{00}\ R_{\psi}\ S_{01}$, the arc from S_{01} to S_{00} represents $S_{01}\ R_{\psi}\ S_{00}$, and so on. Furthermore $S_{00}\ R_{\psi}\ S_{00}$, $S_{01}\ R_{\psi}\ S_{01}$, $S_{10}\ R_{\psi}\ S_{10}$ and $S_{11}\ R_{\psi}\ S_{11}$, these correspondences being shown on the graph as loops.

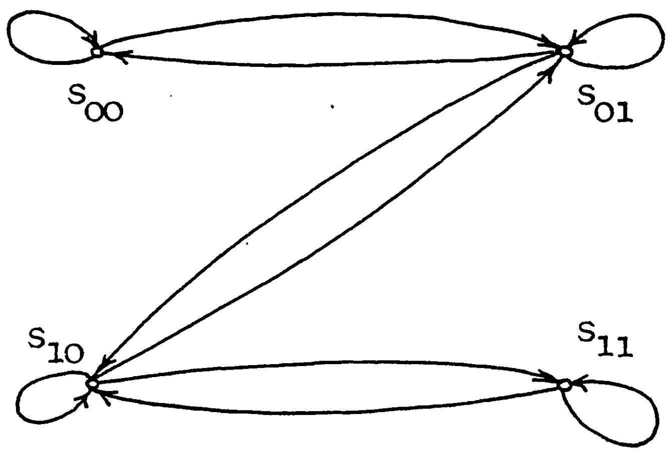


Figure 1.3 The compatibility relation R_{ψ} associated with cover $\psi = (S_{00}\ S_{01})\ (S_{01}\ S_{10})\ (S_{10}\ S_{11})$

Reflexive symmetric relations are called "compatibility" relations, and the above shows that the cover ψ defines an associated compatibility relation R_{ψ}.

Furthermore it has been shown that cover ψ is preserved, so the associated relation R_ψ is of particular interest, and this is shown in figure 1.4(a). The figure shows (from figure 1.3) that R_ψ relates S_{10} to S_{01}, and shows (from figure 1.1) that these states have respective 0-successors S_{01} and S_{00}. Furthermore these successors are themselves related by R_ψ, that is $S_{01}\ R_\psi\ S_{00}$, and this can be verified from figure 1.3. In a sense the relationship expressed as R_ψ passes from the states S_{10}, S_{01} to their successors S_{01} and S_{00}, and this property is not limited to the states S_{10} and S_{01}, as figure 1.4(b) illustrates.

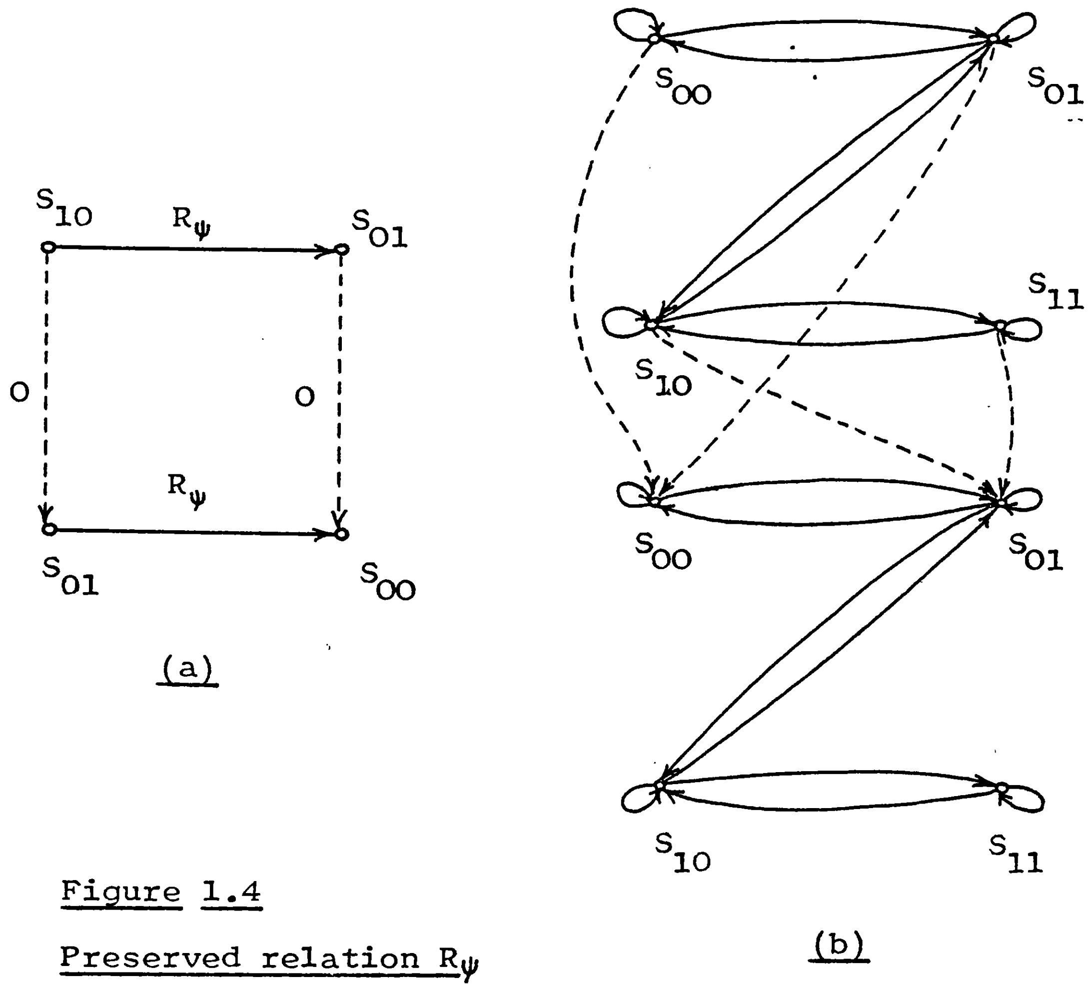


Figure 1.4

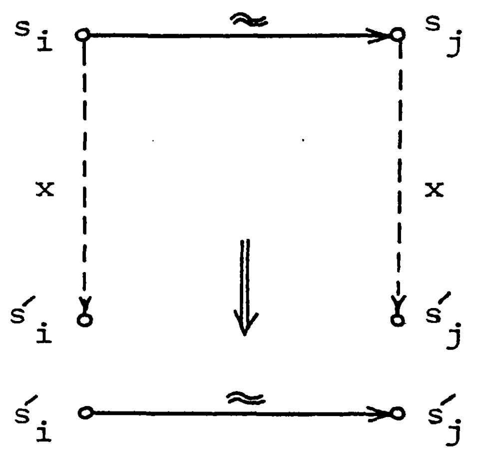
Preserved relation R_ψ

Any relationship $s_i \; R_\psi \; s_j$ induces a relationship $s'_i \; R_\psi \; s'_j$, where s'_i and s'_j are the respective 0-successors of the states s_i and s_j from S, so relation R_ψ is "preserved" under the state transitions for input 0. Similarly, the relation R_ψ is preserved for input 1, so the relation is preserved under all state transitions and can be said to be preserved "within" transition system T. In effect the relation R_ψ is "mapped into itself" under the state transitions, and figure 1.5 shows that this occurs since cover ψ is preserved. Here s_i and s_j represent states of the transition system T, and the figure shows that for input x, which can be taken as either input 0 or input 1, s'_i is the successor of s_i and s'_j is the successor of s_j. Since $s_i \; R_\psi \; s_j$ state s_i must belong to some cover block containing s_j, so the image of this block for input x must contain both s'_i and s'_j. Furthermore, since ψ is a preserved cover, the image of this block must be a subset of at least one block of cover ψ , therefore some cover block must contain s'_i and s'_j, in which case $s'_i \; R_\psi \; s'_j$. This argument is expressed by the implication of figure 1.5.

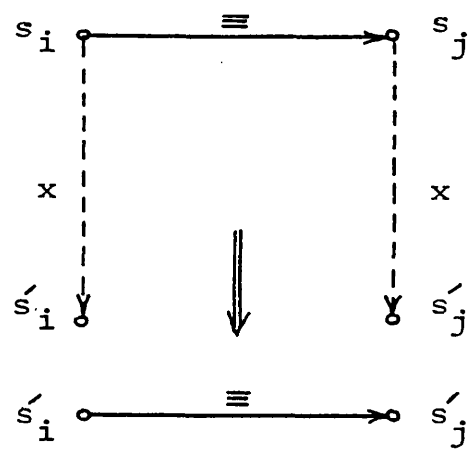
$$\begin{tikzcd}
s_i \arrow[r, "R_\psi"] \arrow[d, dashed, "x"'] & s_j \arrow[d, dashed, "x"] \\
s'_i \arrow[r, phantom, "\Downarrow"] & s'_j \\
s'_i \arrow[r, "R_\psi"] & s'_j
\end{tikzcd}$$

Figure 1.5 Preserved relation R_ψ

Such preserved relations are encountered throughout finite-automata theory, for example the relation of state-compatibility [Booth] is preserved under state transitions, and this is shown in figure 1.6(a). Here s_i and s_j are compatible states of an automaton, that is $s_i \approx s_j$, and for some input x these states have respective successors s'_i and s'_j, so these successors must be compatible, that is $s'_i \approx s'_j$.



(a) State compatibility



(b) State equivalence

Figure 1.6

Similarly the relation of state-equivalence [Booth] over the state-set of an automaton is preserved, and this is shown in figure 1.6(b). If s_i and s_j are equivalent states, that is if $s_i \equiv s_j$, and for some input x these states have respective successors s'_i and s'_j, then these successors must be equivalent, that is $s'_i \equiv s'_j$. There is, however, an important distinction to be made between state-equivalence and state-compatibility. State-equivalence is reflexive, symmetric and transitive, and is therefore an "equivalence" over the automaton state-set. Consequently state-equivalence is a preserved equivalence relation, that is state-equivalence is a "congruence".

Reconsidering now the preserved cover ψ and the associated compatibility relation R_ψ, it has been shown that relation R_ψ is preserved since ψ is a preserved cover, and in fact any preserved cover defines a preserved compatibility relation. Conversely a given preserved compatibility relation defines a preserved cover, and this can be appreciated by observing that the blocks of cover $\psi = (S_{00}\ S_{01})(S_{01}\ S_{10})\ (S_{10}\ S_{11})$ can be reconstructed from figure 1.3, as shown in figure 1.7.

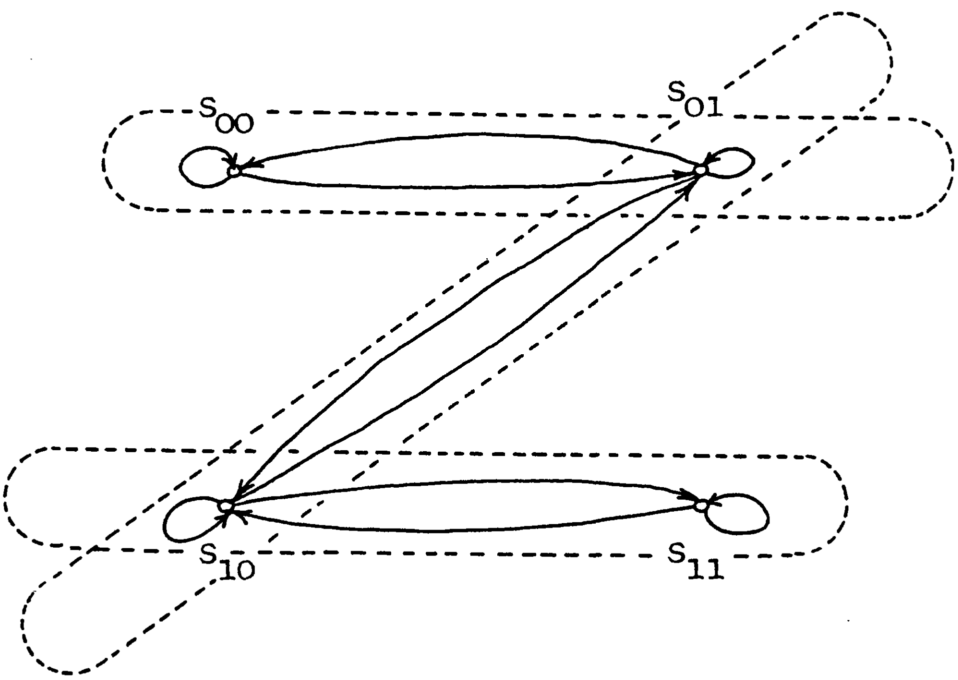


Figure 1.7 The graph of relation R_ψ, with the blocks of cover ψ represented as fully-connected subgraphs

The blocks of cover ψ form maximal fully-connected subgraphs on the graph of relation R_ψ, where by a "fully-connected" subgraph it is meant that any two nodes within the subgraph are related. For example S_{00} and S_{01} form a fully-connected subgraph since figure 1.7 shows $S_{00}\ R_\psi\ S_{01}$, $S_{01}\ R_\psi\ S_{00}$, $S_{00}\ R_\psi\ S_{00}$ and $S_{01}\ R_\psi\ S_{01}$, furthermore this fully-connected subgraph is "maximal" since the subgraph does not lie completely within any other fully-connected subgraph. Consequently the graph of relation R_ψ expresses that $(S_{00}\ S_{01})$ is a block of the associated cover, and in the same way figure 1.7 shows that $(S_{01}\ S_{10})$ and $(S_{10}\ S_{11})$ are blocks, giving the cover $\psi = (S_{00}\ S_{01})\ (S_{01}\ S_{10})\ (S_{10}\ S_{11})$.

Clearly the relationship between cover blocks and maximal fully-connected subgraphs is the key to constructing a cover from a given compatibility relation, and it is important to establish this more generally. Any cover block will form a "complete polygon" on the graph of the associated compatibility relation, that is each block will form a polygon with each vertex related to all the others. In the case of a "nondegenerate" cover, where no cover block is a subset of another, the complete polygons representing the cover blocks will be "maximal" in the sense that none of these polygons will lie completely within another. Conversely a nondegenerate cover can be derived from any given compatibility relation by finding the maximal complete polygons on the graph of the relation, and forming the corresponding cover blocks. More formally a specific nondegenerate cover ψ_R can be derived from any given compatibility relation R by finding the "maximal R-classes" associated with the relation, where by a R-class is meant a set B where s_i R s_j for any s_i and s_j in B. The R-classes correspond to the complete polygons on the graph of relation R, and a R-class B is "maximal" if B is not a subset of any other R-class, so the maximal R-classes correspond to the maximal complete polygons. Then the nondegenerate cover ψ_R corresponding to the compatibility relation R is the set of the maximal R-classes.

Furthermore if R is a compatibility relation over an automaton state-set, and R is preserved under state transitions, then the associated cover ψ_R will be a preserved cover. To appreciate this let B be some block of the cover ψ_R , let x be some input and let B' denote the image of the set B for input x, so B' is the set formed by the x-successors of the states within B. If a'_i and a'_j are arbitrary states from the set B' then a'_i must be the x-successor of some state a_i from B, and a'_j must be the x-successor of some state a_j from B. Since B is a block of cover ψ_R the block B is a (maximal) R-class, and from above the R-class B contains a_i and a_j so a_i and a_j must be related, that is a_i R a_j. Therefore a_i R a_j where a_i has x-successor a'_i and a_j has x-successor a'_j, and relation R is preserved so the relation must pass from a_i and a_j to their successors, giving a'_i R a'_j. This shows that any elements a'_i and a'_j from the set B' are related, in which case the image B' of cover block B is itself a R-class. Then B' must either be a maximal R-class, in which case B' will form a block of cover ψ_R, or B' must be a subset of some maximal R-class, and ψ_R is the set of all the maximal R-classes so there must exist some block of cover ψ with B' as a subset. Hence the image of any block of the cover ψ_R must be a subset of at least one cover block, and this confirms that the cover ψ_R associated with the preserved compatibility relation R is a preserved cover.

Any preserved compatibility relation defines a preserved cover, and this means that the graph of a preserved compatibility relation will map into itself, as has been illustrated in figure 1.4(b). If a compatibility relation R over the state-set of an automaton is preserved under state transitions, and B is any block of the associated cover ψ_R, then B will be represented as a maximal complete polygon and the image B' of B for any input x will be a complete polygon but will not necessarily be maximal. However maximal complete polygons will always be assigned onto complete polygons, just as in figure 1.4(b) the maximal complete polygon formed by S_{01} and S_{10} is assigned onto the (maximal) complete polygon formed by S_{00} and S_{01}, and the maximal complete polygon formed by S_{00} and S_{01} is assigned onto the "degenerate" complete polygon representing $S_{00}\ R_{\psi}\ S_{00}$, and is assigned "into" the maximal complete polygon formed by S_{00} and S_{01}. Any preserved compatibility relation can be visualised as an assignment of maximal complete polygons into maximal complete polygons on the graph of the relation, and the relation of state-compatibility provides an important example. It has been observed that state-compatibility is preserved under the state transitions of an automaton, so the associated cover $\psi_{\approx}$ of the automaton state-set, the "final-class" of the automaton, is a preserved cover, and can be determined by forming maximal complete polygons on the graph of the state-compatibility relation [Kohavi]. Similarly the relation of state-equivalence defines a preserved "equivalence partition", since the maximal complete polygons associated with any equivalence relation are disjoint.

Preserved covers are often encountered as representations of preserved compatibility relations, however preserved covers are important in an additional sense, since each preserved cover can be used to define at least one transition system closely related to the "parent". For example figure 1.2(b) shows that the block $(S_{01}\ S_{10})$ of preserved cover ψ has O-image $\{S_{00}\ S_{01}\}$, so in a sense input O assigns block $(S_{01}\ S_{10})$ to block $(S_{00}\ S_{01})$. Furthermore, since the cover ψ is preserved there is always at least one way of assigning a given block, for a given input, to a cover block with the image of the given block as a subset, and these assignments produce a "transition system" with cover blocks instead of states. The relevant details are given in the table, for example block $(S_{00}\ S_{01})$ of cover ψ has O-image $\{S_{00}\}$, and $\{S_{00}\}$ is a subset of cover block $(S_{00}\ S_{01})$. Similarly the O-image of cover block $(S_{01}\ S_{10})$ is a subset (but not a proper subset) of cover block $(S_{00}\ S_{01})$, and the 0-image of cover block $(S_{10}\ S_{11})$ is $\{S_{01}\}$, which is a subset of cover block $(S_{00}\ S_{01})$ and of cover block $(S_{01}\ S_{10})$.

	0	1
$(S_{00}\ S_{01})$	$\{S_{00}\} \subseteq (S_{00}\ S_{01})$	$\{S_{10}\} \subseteq (S_{10}\ S_{11})$, $\subseteq (S_{01}\ S_{10})$
$(S_{01}\ S_{10})$	$\{S_{00}\ S_{01}\} \subseteq (S_{00}\ S_{01})$	$\{S_{10}\ S_{11}\} \subseteq (S_{10}\ S_{11})$
$(S_{10}\ S_{11})$	$\{S_{01}\} \subseteq (S_{00}\ S_{01})$, $\subseteq (S_{01}\ S_{10})$	$\{S_{11}\} \subseteq (S_{10}\ S_{11})$

Preserved cover $\psi = (S_{00}\ S_{01})\ (S_{01}\ S_{10})\ (S_{10}\ S_{11})$

The aim is to assign each cover block to a cover block having the image as a subset, so for input 0 the block $(S_{00}\ S_{01})$ must be assigned to block $(S_{00}\ S_{01})$, and block $(S_{01}\ S_{10})$ must also be assigned to block $(S_{00}\ S_{01})$. However block $(S_{10}\ S_{11})$ can be assigned either to block $(S_{00}\ S_{01})$ or to block $(S_{01}\ S_{10})$, since either of these blocks has the image $\{S_{01}\}$ as a subset, and similarly for input 1 the block $(S_{00}\ S_{01})$ can be assigned to block $(S_{10}\ S_{11})$ or to block $(S_{01}\ S_{10})$. Consequently there are four possible systems of assignments, and one of these is illustrated in figure 1.8.

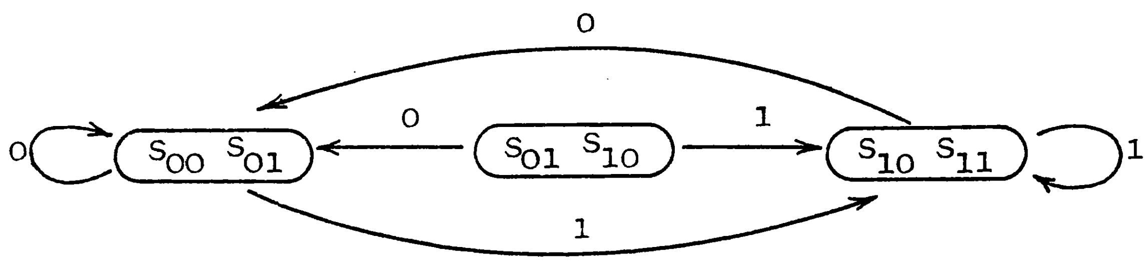


Figure 1.8 Image system T/ψ

For example the arc labelled 0 from block (S_{10} S_{11}) to block (S_{00} S_{01}) shows that block (S_{10} S_{11}) has been assigned to block (S_{00} S_{01}) for input 0, and similarly block (S_{00} S_{01}) has been assigned to block (S_{10} S_{11}) for input 1. Such a system of assignments will be called an "image" system, and the original transition system T will be called the "parent", furthermore the image system of figure 1.8 has been denoted T/ψ to show that this image of T is based on the preserved cover ψ. Comparison with figure 1.1 suggests that image T/ψ is closely related to transition system T, and the study of this relationship introduces the idea of a "weak homomorphism".

A "homomorphism" is a mapping of a given algebra into another so that the "structure" of the given algebra, but not necessarily the detail, is preserved. The homomorphism concept is important throughout abstract algebra, and is encountered for example in the study of quotient groups [Fraleigh]. The parent group is mapped onto the quotient group by a natural homomorphism and therefore the structure of the parent group, but not the detail, is evident from the quotient group.

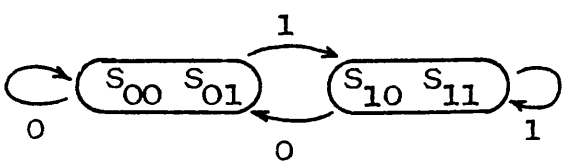
For a more immediate example of a homomorphism consider the parent transition system T of figure 1.1, as expressed in the table of figure 1.9(a). From the table it is evident that the state transitions are not random, indeed it might be said that the transition system is "structured", meaning that the table can be processed to give table (b). Here blocks (S_{00} S_{01}) and (S_{10} S_{11}) have been formed, and the images of these blocks are represented within the table. For example the 0-image of block (S_{00} S_{01}) is $\{S_{00}\}$, and the 0-image of block (S_{10} S_{11}) is $\{S_{01}\}$.

	0	1
S_{00}	S_{00}	S_{10}
S_{01}	S_{00}	S_{10}
S_{10}	S_{01}	S_{11}
S_{11}	S_{01}	S_{11}

Table (a)

	0	1
S_{00} S_{01}	S_{00} S_{00}	S_{10} S_{10}
S_{10} S_{11}	S_{01} S_{01}	S_{11} S_{11}

Table (b)



(c) Image T/π

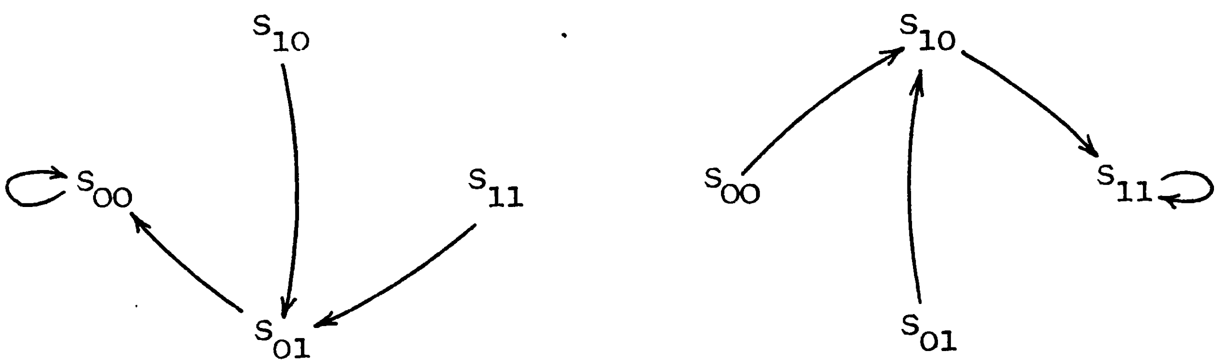
	0	1
(S_{00} S_{01})	(S_{00} S_{01})	(S_{10} S_{11})
(S_{10} S_{11})	(S_{00} S_{01})	(S_{10} S_{11})

Table (d)

Figure 1.9

Clearly the image of any block for any input is a subset of some block, so $\pi = (S_{00}\ S_{01})(S_{10}\ S_{11})$ is a "preserved partition" of the transition system, and can be used to define an image just as four images of transition system T could be derived using the preserved cover ψ. In the case of a preserved partition, however, the image of a given block for a given input must be a subset of just one block, so a preserved partition defines just one image system, and in particular the preserved partition π defines the image system T/π of figure 1.9(c). For example the arc labelled 1 from block $(S_{00}\ S_{01})$ to block $(S_{10}\ S_{11})$ expresses that the image of block $(S_{00}\ S_{01})$ for input 1 is a subset of block $(S_{10}\ S_{11})$, and this can be confirmed from table (b) since the 1-image of block $(S_{00}\ S_{01})$ is $\{S_{10}\}$. Furthermore the image system T/π can be represented as in table (d), and the "structure" of table (a), in the sense that the table can be partitioned to give table (b), is preserved in table (d) but the detail is lost. For example it is evident from table (d), since the table shows that the 0-image of block $(S_{00}\ S_{01})$ is a subset of block $(S_{00}\ S_{01})$, that the 0-successor of state S_{00} must be either S_{00} or S_{01}, but it is not possible to be more specific. This illustrates that the image system T/π is an image of transition system T under a structure preserving mapping, that is T/π is an image of T under a "homomorphism". The aim is to investigate this homomorphism more closely, however it is useful first to express the systems T and T/π as unary algebras.

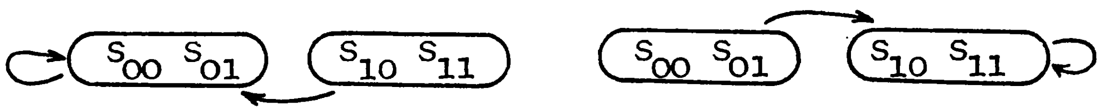
With any given state of transition system T and any given input is associated a successor state, for example figure 1.1 shows that successor state S_{10} is associated with state S_{00} and input 1. Consequently, the state transitions can be formalised as a mapping from Cartesian product SxX to state-set S. With equal validity however the state transitions can be formalised as mappings $\bar{0}$ and $\bar{1}$ over S, and these mappings, associated with the respective inputs 0 and 1, are shown in figure 1.10. For example figure 1.1 shows that the states S_{00}, S_{01}, S_{10} and S_{11} have respective 0-successors S_{00}, S_{00}, S_{01} and S_{01}, and this gives the mapping $\bar{0}$ of figure 1.10(a).



(a) Mapping $\bar{0}$ over S (b) Mapping $\bar{1}$ over S

Figure 1.10

The mappings $\bar{0}$ and $\bar{1}$ are mappings from state-set S to state-set S, that is the mappings are "unary" mappings over S, and transition system T can be formalised as $T = \langle S\ \bar{X} \rangle$, where $\bar{X} = \{\bar{0}, \bar{1}\}$ is the set of the unary mappings over S. Then $\langle S\ \bar{X} \rangle$ is a "unary algebra", and in much the same way the image T/π, as shown in figure 1.9(c), can be represented by defining mappings $\bar{0}^{\pi}$ and $\bar{1}^{\pi}$ over partition π, as shown in figure 1.11.

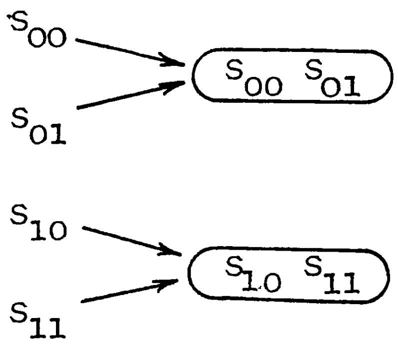


(a) Mapping $\bar{0}^{\pi}$ over π (b) Mapping $\bar{1}^{\pi}$ over π

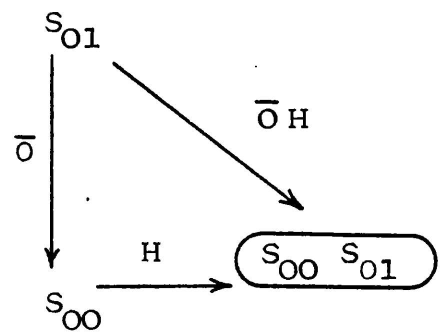
Figure 1.11

For example figure 1.9(c) shows that block (S_{10} S_{11}) is assigned to block (S_{00} S_{01}) for input 0, and shows that block (S_{00} S_{01}) is assigned to itself, so the mapping $\bar{0}^{\pi}$ expressing these assignments is that of figure 1.11(a). Similarly, consideration of figure 1.9(c) for input 1 gives the mapping $\bar{1}^{\pi}$, and the image system T/π can be formalised as the unary algebra $T/\pi = \langle \pi\ \bar{X}_{\pi} \rangle$ where $\bar{X}_{\pi} = \{\bar{0}^{\pi}, \bar{1}^{\pi}\}$ is the set of the unary mappings over π.

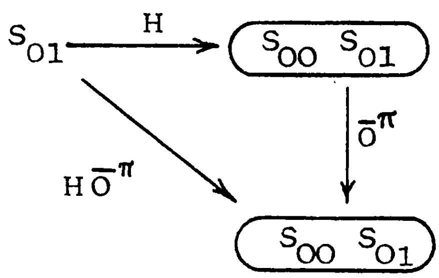
It has been suggested that image system T/π is an image of the parent transition system T under a structure-preserving mapping or "homomorphism", and in fact the homomorphism relating T to T/π is the natural or "canonical" mapping H from S to π, as shown in figure 1.12(a). This mapping relates each element of state-set S to the block of partition π containing this element, and the "structure preserving" property of this mapping can be appreciated by considering the interaction of the mappings $\bar{O}$, $\bar{O}^{\pi}$ and H. Figure 1.12(b) shows state S_{01} of the parent transition system and shows, in accordance with figure 1.10(a), that state S_{01} is assigned by mapping $\bar{O}$ to state S_{00}.



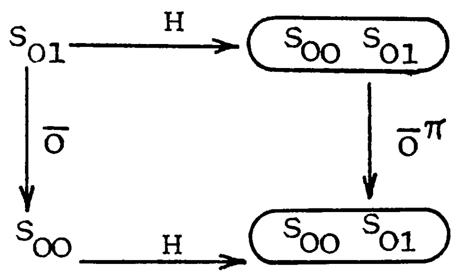
(a) Mapping $H: S \longrightarrow \pi$



(b) Assignment $\bar{O}H$



(c) Assignment $H\bar{O}^{\pi}$



(d) Commutative graph

Figure 1.12

Furthermore figure 1.12(a) shows that mapping H assigns state S_{00} to the block (S_{00} S_{01}) of partition π, so a "composite" assignment $\bar{O}H$ can be conceived so that $\bar{O}H$ expresses the effect of assignment $\bar{O}$ followed by assignment H. In particular $\bar{O}$ assigns state S_{01} to state S_{00}, and H assigns S_{00} to block (S_{00} S_{01}) so $\bar{O}H$ assigns S_{01} to (S_{00} S_{01}), as shown in figure 1.12(b).

Considering now figure 1.12(c), here it is shown that mapping H assigns state S_{01} to block (S_{00} S_{01}) of partition π, and figure 1.11(a) shows that $\bar{O}^{\pi}$ assigns block (S_{00} S_{01}) to block (S_{00} S_{01}). Consequently the composite assignment $H\bar{O}^{\pi}$ assigns state S_{01} to block (S_{00} S_{01}), and it is apparent from the figures (b) and (c) that the assignments $H\bar{O}^{\pi}$ and $\bar{O}H$ have the same effect, so that in each case S_{01} is assigned to block (S_{00} S_{01}). Consequently the graphs can be combined to give graph (d), and this graph is "commutative" in the sense that directed paths from any node have a common termination.

Furthermore this commutative property is not confined to state S_{01}, so that a similar commutative graph can be formed for each of the states S_{00}, S_{01}, S_{10} and S_{11} from S, and figure 1.13 represents a superimposition of the independent commutative graphs. Clearly the mappings $\bar{O}$, $\bar{O}^{\pi}$ and H interact in a systematic way, and this interaction can be expressed as the identity $\bar{O}H = H\bar{O}^{\pi}$, meaning that the composite assignments $\bar{O}H$ and

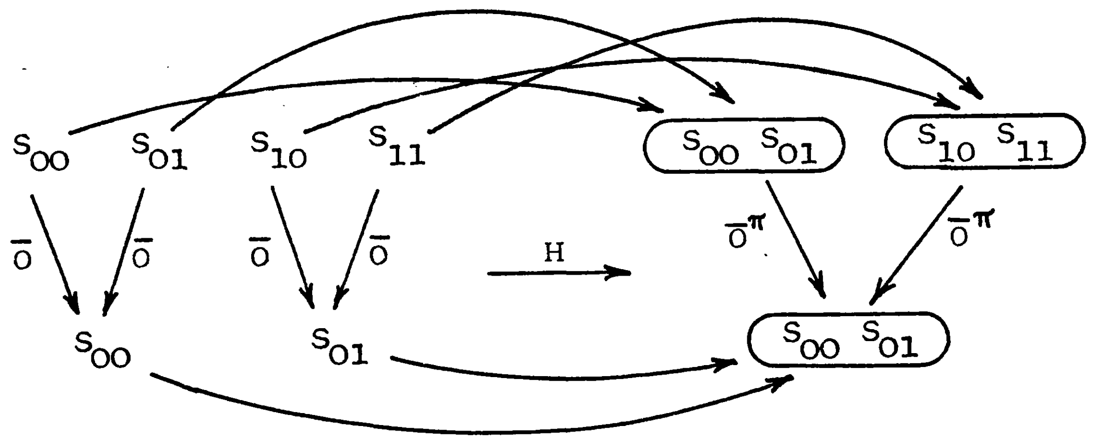
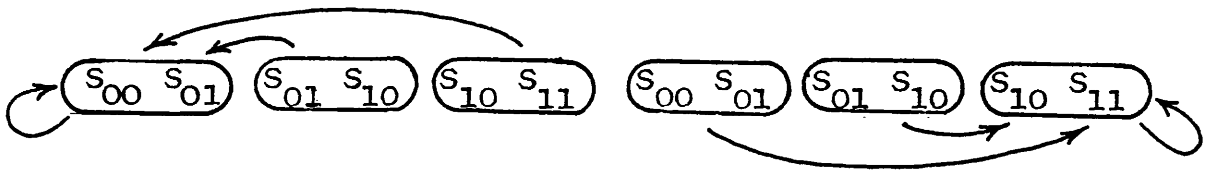


Figure 1.13 The interactive mappings O, $\bar{O}^{\pi}$ and H

$H\bar{O}^{\pi}$ always have the same effect. In a sense mapping H formalises a "factorisation" of mapping $\bar{O}$ to produce mapping $\bar{O}^{\pi}$, or expresses the "structure" of mapping $\bar{O}$ as the mapping $\bar{O}^{\pi}$. In much the same way the assignments $\bar{I}H$ and $H\bar{I}^{\pi}$ are identical, and the identities $\bar{O}H = H\bar{O}^{\pi}$, $\bar{I}H = H\bar{I}^{\pi}$ express that mapping H is a homomorphism of algebra $T = \langle S\ \bar{X} \rangle$ to the algebra $T/\pi = \langle \pi\ \bar{X}_{\pi} \rangle$, so that T/π is a "homomorphic image" of T.

The image system associated with any preserved partition is always a homomorphic image of the parent transition system, just as a quotient group is a homomorphic image of the parent group [Fraleigh]. However the relationship between a transition system and an image based on a preserved cover is more complex, and this can be considered once the previously derived image system T/ψ is expressed as a unary algebra. From figure 1.8 each of the blocks $(S_{00}\ S_{01})$, $(S_{01}\ S_{10})$ and $(S_{10}\ S_{11})$ of cover ψ is assigned to block $(S_{00}\ S_{01})$ for input O, and this can be expressed as the mapping $\bar{O}^{\psi}$ over ψ as shown in figure 1.14(a). Similarly each of these blocks is assigned to block (S_{10} S_{11}) for input 1, giving the mapping $\overline{1}^{\psi}$ over ψ as in figure 1.14(b), and then image T/ψ can be expressed as the unary algebra $T/\psi = \langle \psi \ \overline{X}_{\psi} \rangle$, where $\overline{X}_{\psi} = \{\overline{0}^{\psi}, \overline{1}^{\psi}\}$ is the set of these mappings over ψ.



(a) Mapping $\overline{0}^{\psi}$ over ψ (b) Mapping $\overline{1}^{\psi}$ over ψ

Figure 1.14

Considering now the canonical relationship between S and ψ, the relation will be denoted H' and is defined so that H' relates each state from $S = \{S_{00}\ S_{01}\ S_{10}\ S_{11}\}$ to each cover block containing this state. This gives the relation H' of figure 1.15, for example H' relates state S_{00} to block (S_{00} S_{01}) and relates state S_{01} to block (S_{00} S_{01}) and to block (S_{01} S_{10}), since each of these blocks contain this state.

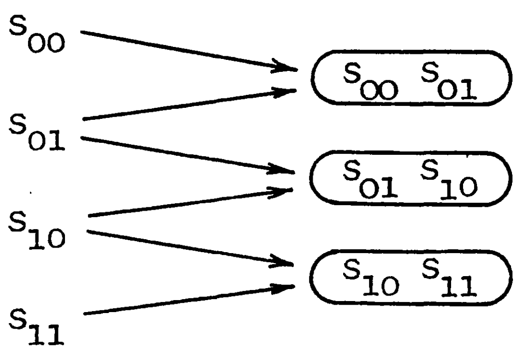


Figure 1.15 Relation H' from S to ψ

Clearly relation H' is "many-to-many" in nature, whereas the canonical relation H from S to partition π was many-to-one, and it is also important to observe that the assignments $\bar{O}H'$ and $H'\bar{O}^{\psi}$ are not identical, that is the assignments do not always have the same effect. This is evident from figure 1.16, for example mapping $\bar{O}$ assigns state S_{10} to state S_{01}, and H' relates S_{01} to block $(S_{01}\ S_{10})$, whereas S_{10} is related by H' to block $(S_{10}\ S_{11})$ and $\bar{O}^{\psi}$ assigns this block to $(S_{00}\ S_{01})$.

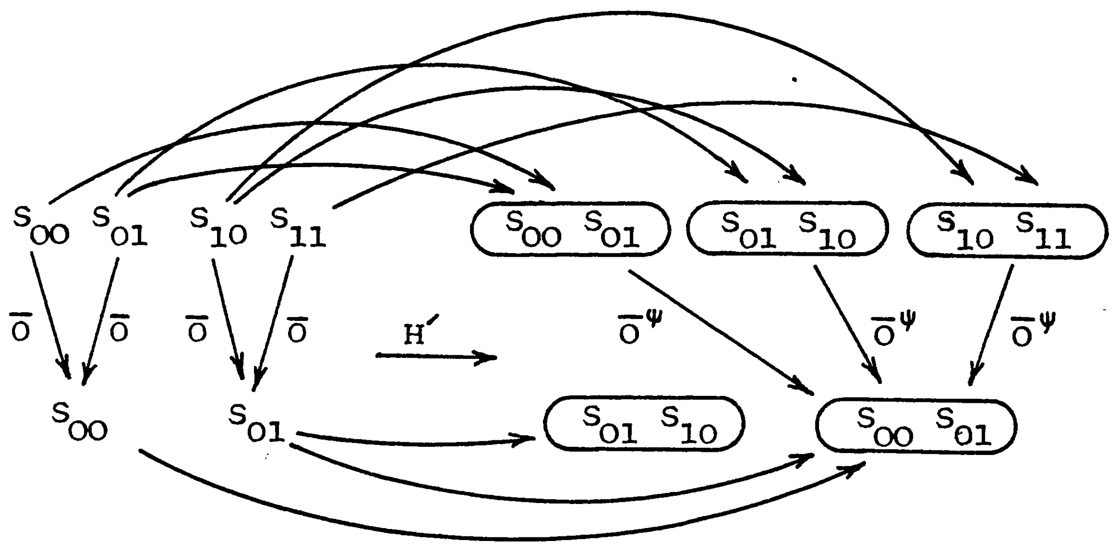


Figure 1.16 Mapping $\bar{O}$, mapping $\bar{O}^{\psi}$ and relation H'

Similarly the assignment $\bar{I}H'$ is not identical to assignment $H'\bar{I}^{\psi}$, so H' is not a "homomorphic" relation from T to T/ψ, and yet it is evident that the transition systems T and T/ψ are closely related. In fact this relationship can be expressed using the "converse" of H', rather than the relation H' itself, where by the converse of H' is meant the relation $(H')^{-1}$ as shown in figure 1.17. For example $(H')^{-1}$ relates block $(S_{00}\ S_{01})$ of cover ψ to the states S_{00} and S_{01}, these being the states within this block, and it can be verified that the graph of relation $(H')^{-1}$ can be derived from figure 1.15 by reversing the arrows.

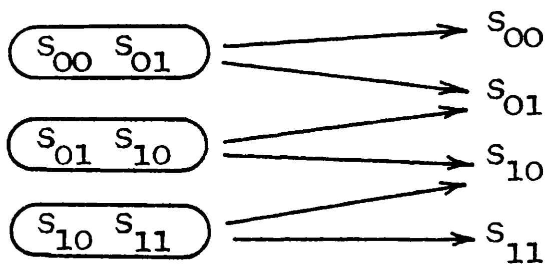
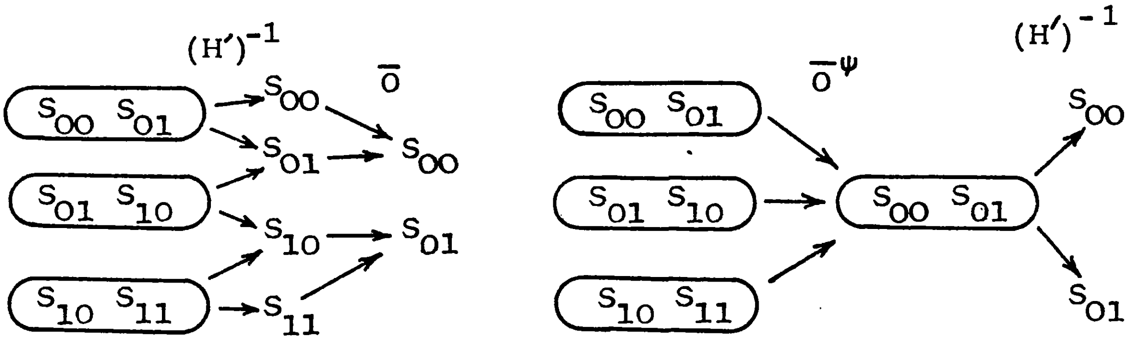


Figure 1.17 Relation $(H')^{-1}$

Since $(H')^{-1}$ relates each block of cover ψ to the corresponding states, and $\bar{0}$ relates each state to a 0-successor, the composite assignment $(H')^{-1}\bar{0}$ relates each block of cover ψ to the 0-successors of the states within the block.

This assignment is shown in figure 1.18(a), for example $(H')^{-1}\bar{0}$ relates block $(S_{01}\ S_{10})$ to state S_{00} since S_{00} is the 0-successor of one of the states within this block, and is in fact the 0-successor of state S_{01}. Similarly the composite assignment $\bar{0}^{\psi}\ (H')^{-1}$ relates each block of ψ to the states within the $\bar{0}^{\psi}$-successor, as shown in figure 1.18(b). For example $\bar{0}^{\psi}(H')^{-1}$ relates block $(S_{01}\ S_{10})$ to state S_{00}, since $\bar{0}^{\psi}$ assigns block $(S_{01}\ S_{10})$ to block $(S_{00}\ S_{01})$, and this block contains S_{00}.



(a) Relation $(H')^{-1}\bar{0}$ (b) Relation $\bar{0}^{\psi}\ (H')^{-1}$

Figure 1.18

These composite assignments are crucial in expressing the way T is related to the "image" system T/ψ , and this can be appreciated by considering figure 1.19. Here x can be taken to be either of the inputs 0 or 1, and in the figure it is assumed that $(H')^{-1}\bar{x}$ relates a block B of cover ψ to a state s.

$$\begin{tikzcd} B \arrow[d, "(H')^{-1}\bar{x}"'] & & B \arrow[d, "\bar{x}^{\psi}(H')^{-1}"] \\ s \arrow[r, phantom, "\Longrightarrow"] & & s \end{tikzcd}$$

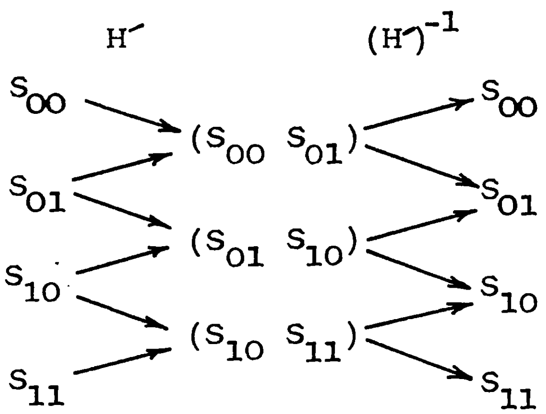
Figure 1.19

Consequently s must be the x-successor of some state within block B, so the image of block B for input x must contain state s. Furthermore mapping $\bar{x}^{\psi}$ assigns block B to some block with the image of block B, for input x, as a subset, therefore the $\bar{x}^{\psi}$ - successor of block B must contain state s, in which case $\bar{x}^{\psi}\ (H')^{-1}$ must assign block B to s. This argument is expressed in figure 1.19 as an implication. so that any relationship under $(H')^{-1}\bar{x}$ must form a relationship under $\bar{x}^{\psi}\ (H')^{-1}$.

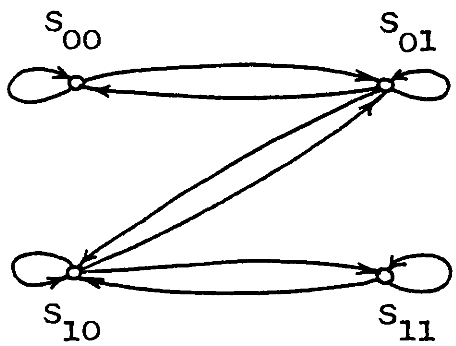
This property can be expressed by the inclusion $(H')^{-1}\bar{x} \subseteq \bar{x}^{\psi}\ (H')^{-1}$, or more specifically $(H')^{-1}\bar{0} \subseteq \bar{0}^{\psi}(H')^{-1}$ and $(H')^{-1}\bar{1} \subseteq \bar{1}^{\psi}(H')^{-1}$, so a relationship under $(H')^{-1}\bar{0}$ must form a relationship under $\bar{0}^{\psi}(H')^{-1}$, and similarly each relationship under $(H')^{-1}\bar{I}$ induces a relationship under $\bar{I}^{\psi}(H')^{-1}$. For example figure 1.18(a) shows that block $(S_{01}\ S_{10})$ is related under $(H')^{-1}\bar{O}$ to state S_{00}, and it can be confirmed from figure 1.18(b) that $\bar{O}^{\psi}\ (H')^{-1}$ relates this block to the same state, indeed it can be confirmed that any relationship under $(H')^{-1}\bar{O}$ is also a relationship under $\bar{O}^{\psi}\ (H')^{-1}$. It is important to note, however, that the converse is invalid, so inclusion cannot be replaced by identity. For example $\bar{O}^{\psi}\ (H')^{-1}$ relates block $(S_{10}\ S_{11})$ to state S_{00}, but this is not a correspondence under $(H')^{-1}\bar{O}$.

The inclusions $(H')^{-1}\bar{O} \subseteq \bar{O}^{\psi}\ (H')^{-1}$ and $(H')^{-1}\bar{I} \subseteq \bar{I}^{\psi}\ (H')^{-1}$ express that H' is a "weak homomorphism" of $T = \langle S\ \bar{X}\rangle$ to $T/\psi = \langle \psi\ \bar{X}_{\psi}\rangle$, in which case the unary algebra T/ψ is an "image" of T under the weak homomorphism H'. In fact each of the image systems associated with a given preserved cover forms an image of the parent system under a weak homomorphism, and the weak homomorphism is the canonical relation associated with the preserved cover. An important example of this is encountered in automaton reduction, since the "reduced" automaton is based on a preserved cover of the nonminimal automaton. Consequently, the state transitions associated with the reduced automaton form an image of the nonminimal automaton under a weak homomorphism.

The preceeding shows that an image system defined from a preserved cover has an associated weak homomorphism, furthermore a weak homomorphism has an associated preserved compatibility relation, and this can be appreciated by considering the weak homomorphism H'. By the "kernel" of H' is meant the relation $H'(H')^{-1}$ as shown in figure 1.20(a), for example $H'(H')^{-1}$ relates S_{00} to S_{01} since S_{00} is contained in block (S_{00} S_{01}), and this block also contains state S_{01}.



(a) Relation $H'(H')^{-1}$, the "kernel" of relation H'



(b) Relation $H'(H')^{-1}$

Figure 1.20

Clearly $H'(H')^{-1}$ relates two states whenever one belongs to a block containing the other. However this is the relation R_ψ from previously, and this can be confirmed by expressing figure 1.20(a) as the relation graph of figure 1.20(b). This shows that $H'(H')^{-1}$ is identical to the compatibility relation R_ψ of figure 1.3, and it has been shown that relation R_ψ is preserved within $T = \langle S\ \overline{X} \rangle$, so it is concluded that the kernel $H'(H')^{-1}$ of weak homomorphism H' is a preserved compatibility relation.

In fact the kernel of any weak homomorphism is a preserved compatibility relation, so it is seen that preserved covers, image systems, weak homomorphism and preserved compatibility relations are closely interrelated. This is illustrated in figure 1.21(a), for example the figure expresses that a preserved cover defines a preserved compatibility relation, and shows that a preserved compatibility relation defines a preserved cover.

$$
\begin{tikzcd}
\text{Preserved cover} \arrow[r, shift left] \arrow[d] & \text{Preserved compatibility relation} \arrow[l, shift left] \\
\text{Image system} \arrow[r] & \text{Weak Homomorphism} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{Preserved partition} \arrow[r, shift left] \arrow[d] & \text{Congruence} \arrow[l, shift left] \\
\text{Quotient system} \arrow[r] & \text{Homomorphism} \arrow[u]
\end{tikzcd}
$$

(a) (b)

Figure 1.21

In particular the preserved cover ψ was used to define the preserved compatibility relation R_ψ of figure 1.3, and the cover was reconstructed from the graph of relation R_ψ by recognising the maximal fully-connected subgraphs, as in figure 1.7. Furthermore a given preserved cover defines at least one image system, for example preserved cover ψ was used to define the image system T/ψ, and figure 1.21(a) also expresses that an image system defines an associated weak homomorphism, this being the canonical relation associated with the preserved cover. Finally, the kernel of a weak homomorphism is a preserved compatibility relation, so a weak homomorphism defines a preserved compatibility relation and therefore defines an associated preserved cover.

This diagram is the basis of a unified appreciation of finite-automata theory, since it has been suggested that preserved covers, image systems, weak homomorphism and preserved relations are important and recurring themes, and the diagram shows that these concepts are closely interrelated. Specifically, the diagram can be used to appreciate the significance of preserved covers in the theory of finite automata. Preserved covers can be used to represent preserved compatibility relations, for example the "final class" of an automaton is a preserved cover representing the state-compatibility relation. Furthermore, preserved covers can be used in forming weak-homomorphic images. Images of a given transition system under weak homomorphism can be used in automaton decomposition, and can be used in forming composite realisations of an automaton using stock units. Then it is especially important to appreciate that a weak homomorphism defines an associated preserved cover, and defines an associated image system. This means that the weak-homomorphic images of a given transition system are represented by the "image systems" based on the preserved covers.

It is also important to consider the way the "homomorphism" concept is related to weak homomorphism. Every homomorphism is a weak homomorphism, so homomorphism is a more "refined" concept, and replaces weak homomorphism whenever preserved partitions and congruences, rather than preserved covers and preserved compatibility relations, are involved. This is shown in figure 1.21(b), which expresses that every preserved partition defines a congruence, and that conversely a given congruence defines a preserved partition. Furthermore a given preserved partition defines a unique image system, and this has been called a "quotient system" in figure 1.21(b) since this idea is encountered in connection with groups, for example, and a "quotient group" is defined [Fraleigh]. More specifically the preserved partition π was used to define T/π as in figure 1.9(c), so T/π is the quotient system based on preserved partition π. Figure 1.21(b) also expresses that a given quotient system defines a homomorphism, for example the canonical relation H from S to π was shown to be a homomorphism of T to T/π, and expresses finally that a given homomorphism defines a congruence, this being the kernel of the homomorphism.

This concludes the survey of preserved covers, preserved relations, image systems, homomorphism and weak homomorphism, however the preceeding has also shown that state transitions can be expressed in the form of a unary algebra, and this is of particular significance. Representation as a unary algebra means that state-transition systems can be analysed using the general properties of unary algebras, indeed using universal algebra [Cohn; Gratzer]. By using universal algebra the properties of state-transition systems are related to corresponding properties of groups, rings and other algebras, and the theory of state-transition systems can be developed in close association with modern algebra [Birkhoff & MacLane; Fraleigh], instead of being considered in isolation. Consequently, in developing the "automaton" concept, the conventional representations [Moore; Mealy] will not be adopted.

CHAPTER TWO : Semiautomata and Automata

2.1 Introduction

It has been suggested that the state transitions of a sequential circuit can be expressed as a unary algebra, and that adopting this representation relates automata theory to universal algebra. The present aim is to formalise this representation of the state-transition aspect of sequential circuits, and to develop the "automaton" concept so that the link with universal algebra is retained. It will also be necessary to introduce additional symbology, especially that associated with input sequences or "tapes". Furthermore the concepts and symbolism from set theory, as summarised in Appendix A, will now be used freely.

2.2 Semiautomata

In forming a representation for a sequential circuit an "input set" $X = \{x_1, x_2, x_3, \ldots\}$, an "output set" $Z = \{z_1, z_2, z_3, \ldots\}$ and a "state set" $S = \{s_1, s_2, s_3, \ldots\}$ can be defined, so that each element of input set X uniquely represents one of the input codes associated with the circuit, each element of the set Z represents an output code and each element of the set S represents a memory code. Considering now the operation of the sequential circuit, the subsequent memory code or "state" of the circuit is determined by the existing combination of the memory code and the applied input code. Hence input symbol x_1 will associate some successor-state $s_1' \in S$ with state s_1, in the sense that the input code represented as x_1 causes a transition from the state represented as s_1 to the state represented as s'_1. Similarly input x_1 associates some successor-state $s'_2 \in S$ with state s_2, associates a successor-state $s'_3 \in S$ with state s_3, and so on, and these associations can be expressed as a set $\{\langle s_1\ s'_1\rangle\ \langle s_2\ s'_2\rangle\ \langle s_3\ s'_3\rangle\ \ldots\}$. To retain the link with input symbol x_1 the set will be denoted $\bar{x}_1$, and it is evident that each element of set S will be assigned to a unique successor-state, so $\bar{x}_1 = \{\langle s_1\ s'_1\rangle\ \langle s_2\ s'_2\rangle\ \langle s_3\ s'_3\rangle \ldots\}$ is a mapping over S and has domain $D[\bar{x}_1] = S$. Similarly a mapping $\bar{x}_2$ over S is associated with input symbol x_2, a mapping $\bar{x}_3$ over S is associated with input x_3, and so on, giving a set $\bar{X} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots\}$ of mappings over the state set S.

To every element $x_i \in X$ corresponds a mapping $\bar{x}_i \in \bar{X}$, so that the set $\bar{X}$ is "indexed" by input set X, in other words a natural mapping or "family" relates X to $\bar{X}$. The idea of a mapping as a family [Halmos] is useful whenever the codomain of the mapping is more important than the mapping itself, for example here it is desired to convey that to each $x_i \in X$ corresponds a unique element $\bar{x}_i \in \bar{X}$, but it is not particularly desired to formalise this correspondence as a mapping F from X to $\bar{X}$ where $\langle x_i\ \bar{x}_i\rangle \in F$. Instead the correspondence is conveyed by the symbolism, so that to $x_i, x_j, x_k, \ldots \in X$ correspond respective elements $\bar{x}_i, \bar{x}_j, \bar{x}_k, \ldots \in \bar{X}$. It should be appreciated however that the relationship between X and $\bar{X}$

is not in general injective, since distinct elements $x_i, x_j \in X$ might exist where $\bar{x}_i = \bar{x}_j$. Indeed if $|S| = n$ then $\bar{X}$ cannot have more than n^n elements, this being the number of distinct nonvoid mappings over a set with n elements.

Since $\bar{X}$ is a set of mappings over the set S, the system $\langle S\ \bar{X}\rangle$ representing the state-transitions of a sequential circuit is a unary algebra. Such a unary algebra is now formalised as a "semiautomaton", and subscript notation (subscript A in the definition) is introduced so that several semiautomata can be considered concurrently.

Definition

A $\underline{X_A\text{-semiautomaton over } S_A}$ (where X_A and S_A are nonvoid sets) is a unary algebra $A = \langle S_A\ \bar{X}_A\rangle$, where the set $\bar{X}_A$ of mappings $\bar{x}^A$ over state-set S_A is indexed by input set X_A.

The above set $\bar{X}_A$ will be called the "transition set" of the semiautomaton $A = \langle S_A\ \bar{X}_A\rangle$, and if this semiautomaton represents the action of a sequential circuit each mapping $\bar{x}^A \in \bar{X}_A$ will have the state set S_A as domain. Then the semiautomaton will be "complete", meaning that $\langle S_A\ \bar{X}_A\rangle$ is a complete unary algebra.

Definition

A semiautomaton $A = \langle S_A \; \overline{X}_A \rangle$ is complete iff

$$(\forall x)(x \in X_A \implies D[\bar{x}^A] = S_A)$$

For example the two-stage shift-register was expressed in the preceeding as the semiautomaton $\langle S \; \overline{X} \rangle$, where $X = \{0, 1\}$ and $D[\bar{0}] = S = D[\bar{1}]$, so this semiautomaton is complete. It should also be appreciated that the above implication should be written

$$(\forall x)(x \in X_A, \; \langle x \;\; \bar{x}^A \rangle \in F \implies D'[\bar{x}^A] = S_A)$$

where F is the indexing surjection from X_A to $\overline{X}_A$, however the indexing will always be implicit. Furthermore a comma, rather than the ampersand, will sometimes be used to denote the conjunction of propositions.

Having formalised the semiautomaton concept, definitions can be given for preserved covers, preserved relations and weak homomorphism. In the preceeding a "cover" was taken to be a set of subsets of a given set, where each element of the given set is contained in at least one of the subsets. This definition is sometimes inadequate, and it is necessary to formalise the cover concept so that the cover blocks are indexed.

Definition

A S_A-cover is a family $\{B\}_I$ of subsets of S_A, where $\cup\{B\}_I = S_A$.

The indexing can often be disregarded, for example the indexing is unnecessary in formalising the preserved cover concept.

<u>Definition</u>

A S_A-cover π is <u>preserved</u> under a mapping $\bar{x}^A$ iff

$$(\forall B)(B \in \pi \implies (\exists B')(B' \in \pi \ \& \ (B)\bar{x}^A \subseteq B'))$$

Considering now preserved relations, the idea has been illustrated in the figures 1.5 and 1.6. To say a relation is "preserved" under a mapping $\bar{x}^A$ means that the relationship passes from related elements to their $\bar{x}^A$-successors.

<u>Definition</u>

A relation R is <u>preserved</u> under a mapping $\bar{x}^A$ iff

$$(\forall a_i)(\forall a_j)(\langle a_i \ a_j \rangle \in R, \ \langle a_i \ a_i' \rangle \in \bar{x}^A, \ \langle a_j \ a_j' \rangle \in \bar{x}^A \implies \langle a_i' \ a_j' \rangle \in R)$$

If a relation R over S_A is preserved under all the mappings $\bar{x}^A$ forming the transition set $\bar{X}_A$, the relation is said to be preserved "within" the semiautomaton $A = \langle S_A \ \bar{X}_A \rangle$. Similarly a S_A-cover is preserved "within" semiautomaton $A = \langle S_A \ \bar{X}_A \rangle$, or is a "preserved cover of semiautomaton A", if preserved under all the transition mappings $\bar{x}^A$ from $\bar{X}_A$. An equivalence preserved within a semiautomaton is a "congruence" of the semiautomaton, and corresponds to a preserved partition.

The preceeding has also introduced "weak homomorphism", for example the inclusions $(H')^{-1}\,\overline{O} \subseteq \overline{O}^{\psi}(H')^{-1}$ and $(H')^{-1}\overline{I} \subseteq \overline{I}^{\psi}(H')^{-1}$ confirmed that the relation H' of figure 1.15 is a weak homomorphism of $T = \langle S\ \overline{X}\rangle$ to $T/\psi = \langle \psi\ \overline{X}_{\psi}\rangle$.

Definition

Let $A = \langle S_A\ \overline{X}_A\rangle$, $B = \langle S_B\ \overline{X}_B\rangle$ be semiautomata where $X_A \subseteq X_B$, and let θ be a relation from S_A to S_B where $D[\theta] = S_A$.

Relation θ is a weak homomorphism of semiautomaton A to semiautomaton B, denoted $A \xrightarrow{\theta} B$, iff

$$(\forall x)(x \in X_A \implies \theta^{-1}\,\bar{x}^A \subseteq \bar{x}^B\,\theta^{-1})$$

Several important properties of weak homomorphisms have been established [Yoeli], in particular the composition of weak homomorphisms produces a weak homomorphism.

Result [Yoeli]

If θ is a weak homomorphism of $A = \langle S_A\ \overline{X}_A\rangle$ to $B = \langle S_B\ \overline{X}_B\rangle$, and θ' is a weak homomorphism of B to $C = \langle S_C\ \overline{X}_C\rangle$, then $\theta\theta'$ is a weak homomorphism of A to C.

To appreciate this assume $A \xrightarrow{\theta} B$ and $B \xrightarrow{\theta'} C$ as above, in which case $X_A \subseteq X_B \subseteq X_C$, furthermore $\theta^{-1}\,\bar{x}^A \subseteq \bar{x}^B\,\theta^{-1}$ and $(\theta')^{-1}\,\bar{x}^B \subseteq \bar{x}^C\,(\theta')^{-1}$ for any $x \in X_A$. Assume $x \in X_A$ and assume $\langle c\ a'\rangle \in (\theta\theta')^{-1}\,\bar{x}^A$, so $\langle c\ a\rangle \in (\theta\theta')^{-1}$ and $\langle a\ a'\rangle \in \bar{x}^A$ for some $a \in S_A$. Then

$\langle a\ c\rangle \in \theta\theta'$, so $\langle a\ b\rangle \in \theta$ and $\langle b\ c\rangle \in \theta'$ for some $b \in S_B$, as shown in figure 2.1.

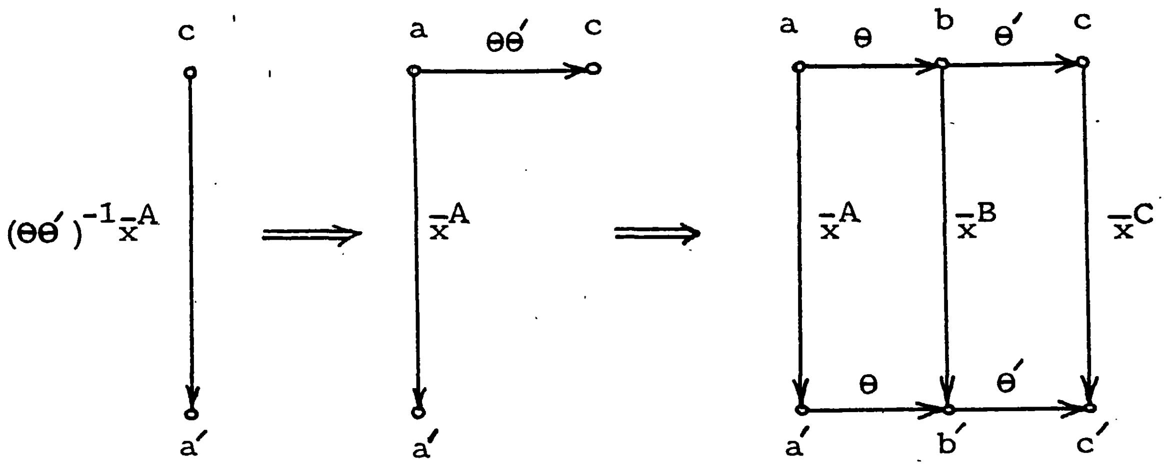


Figure 2.1

$\langle c\ a'\rangle \in (\theta\theta')^{-1}\,\bar{x}^A$ implies $\langle c\ a'\rangle \in \bar{x}^C\,(\theta\theta')^{-1}$

Then $\langle b\ a'\rangle \in \theta^{-1}\,\bar{x}^A$, so $\langle b\ a'\rangle \in \bar{x}^B\,\theta^{-1}$, in which case $\langle b\ b'\rangle \in \bar{x}^B$ and $\langle b'\ a'\rangle \in \theta^{-1}$ for some b'. Consequently $\langle c\ b'\rangle \in (\theta')^{-1}\,\bar{x}^B$, so $\langle c\ b'\rangle \in \bar{x}^C\,(\theta')^{-1}$, and then $\langle c\ c'\rangle \in \bar{x}^C$ and $\langle c'\ b'\rangle \in (\theta')^{-1}$ for some c'. Hence $\langle c'\ b'\rangle \in (\theta')^{-1}$, and $\langle b'\ a'\rangle \in \theta^{-1}$ so $\langle c'\ a'\rangle \in (\theta')^{-1}\,\theta^{-1}$, that is $\langle c'\ a'\rangle \in (\theta\theta')^{-1}$, and $\langle c\ c'\rangle \in \bar{x}^C$ so $\langle c\ a'\rangle \in \bar{x}^C\,(\theta\theta')^{-1}$.

Therefore $\langle c\ a'\rangle \in (\theta\theta')^{-1}\,\bar{x}^A$ implies $\langle c\ a'\rangle \in \bar{x}^C\,(\theta\theta')^{-1}$, and $x \in X_A$ is arbitrary so

$(\forall x)(x \in X_A \implies (\theta\theta')^{-1}\,\bar{x}^A \subseteq \bar{x}^C\,(\theta\theta')^{-1})$,

in addition $D[\theta\theta'] = S_A$, confirming that $\theta\theta'$ is a weak homomorphism of A to C.

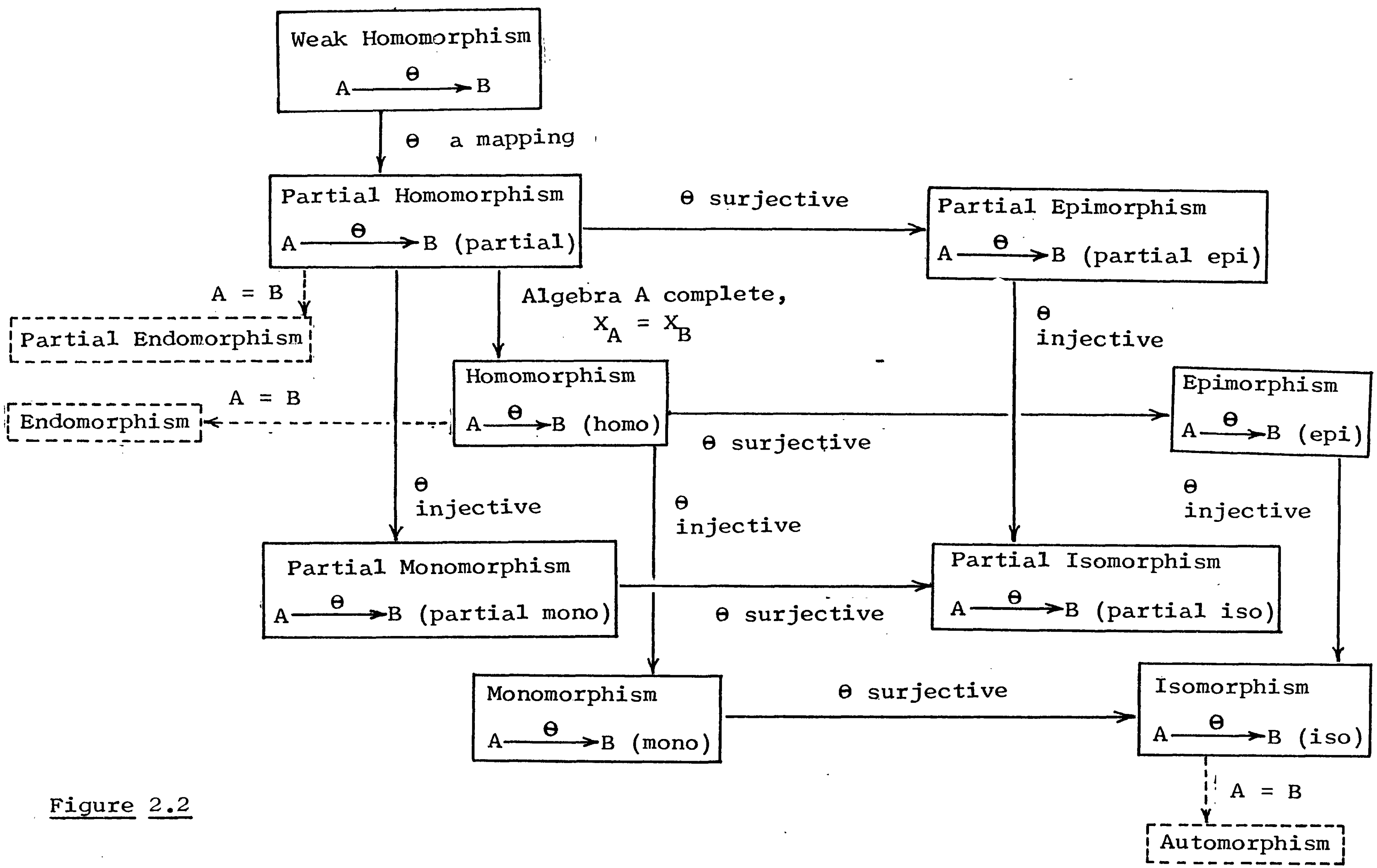

Weak Homomorphism
$A \xrightarrow{\theta} B$
θ a mapping
Partial Homomorphism
$A \xrightarrow{\theta} B$ (partial)
θ surjective
Partial Epimorphism
$A \xrightarrow{\theta} B$ (partial epi)
A = B
Partial Endomorphism
Algebra A complete, $X_A = X_B$
θ injective
Homomorphism
$A \xrightarrow{\theta} B$ (homo)
A = B
Endomorphism
θ surjective
Epimorphism
$A \xrightarrow{\theta} B$ (epi)
θ injective
θ injective
θ injective
Partial Monomorphism
$A \xrightarrow{\theta} B$ (partial mono)
θ surjective
Partial Isomorphism
$A \xrightarrow{\theta} B$ (partial iso)
Monomorphism
$A \xrightarrow{\theta} B$ (mono)
θ surjective
Isomorphism
$A \xrightarrow{\theta} B$ (iso)
A = B
Automorphism

Figure 2.2

It is useful, finally, to establish the variants of the homomorphism concept, and these are shown in figure 2.2. For example if a mapping θ is a weak homomorphism of A to B then θ is a "partial homomorphism" of A to B, and this is written $A \xrightarrow{\theta} B$ (partial). The inclusion $\theta^{-1} \bar{x}^A \subseteq \bar{x}^B \theta^{-1}$ can then be replaced by $\bar{x}^A \theta \subseteq \theta \bar{x}^B$, since if θ is a mapping these inclusions are equivalent. An injective partial homomorphism is a "partial monomorphism", denoted $A \xrightarrow{\theta} B$ (partial mono), a surjective partial homomorphism is a "partial epimorphism" and a bijective partial homomorphism is a "partial isomorphism", respectively $A \xrightarrow{\theta} B$ (partial epi) and $A \xrightarrow{\theta} B$ (partial iso). Figure 2.2 also shows that a partial homomorphism $A \xrightarrow{\theta} B$ (partial), where $X_A = X_B$ and the semiautomaton A is complete, is a homomorphism in the normal sense and this can be written $A \xrightarrow{\theta} B$ (homo). Then the inclusion $\bar{x}^A \theta \subseteq \theta \bar{x}^B$ can be replaced by the equivalent identity $\bar{x}^A \theta = \theta \bar{x}^B$, and θ is termed a "monomorphism" if injective, an "epimorphism" if surjective or an "isomorphism" if bijective.

2.3 Image semiautomata

The idea of an "image system" is an important extension of the semiautomaton concept, such systems being formalised as "factors" [Yoeli] or "image semiautomata".

Definition

A unary algebra $G = \langle S_G \ \overline{X}_G \rangle$ is an image semiautomaton or a π_i-image of a semiautomaton $A = \langle S_A \ \overline{X}_A \rangle$ iff $S_G = \pi_i$ where π_i is a S_A-cover, $X_G = X_A$,

$(\forall B)(\forall x)(B \in \pi_i, x \in X_A, \ (B)\overline{x}^A = \emptyset \Longrightarrow B \notin D[\overline{x}^G])$

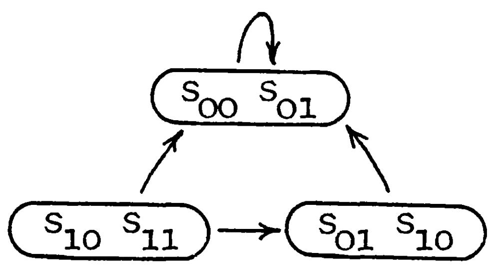
$(\forall B)(\forall x)(B \in \pi_i, x \in X_A, \ (B)\overline{x}^A \neq \emptyset \Longrightarrow (\exists B')(\langle B\, B'\rangle \in \overline{x}^G \ \& \ (B)\overline{x}^A \subseteq B'))$

At least one image semiautomaton can be formed from any given preserved cover, for example consider the way the preserved cover $\psi = (S_{00} \ S_{01})(S_{01} \ S_{10})(S_{10} \ S_{11})$ was used to define the image system $T/\psi = \langle \psi \ \overline{X}_\psi \rangle$. The table on page 14 shows the image of each cover block for each input, and illustrates that each such image is included in a least one cover block. More formally, the table represents relations $\overline{0}^{\mathcal{R}}$ and $\overline{1}^{\mathcal{R}}$ over ψ where

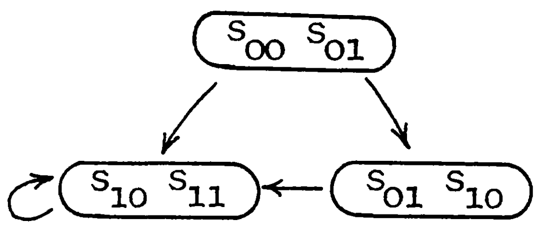
$\overline{0}^{\mathcal{R}} = \{\langle B \ B'\rangle \mid B, B' \in \psi, \ (B)\overline{0} \neq \emptyset \ \& \ (B)\overline{0} \subseteq B'\}$ and

$\overline{1}^{\mathcal{R}} = \{\langle B \ B'\rangle \mid B, B' \in \psi, \ (B)\overline{1} \neq \emptyset \ \& \ (B)\overline{1} \subseteq B'\}$,

these relations being shown in figure 2.3.



(a) Relation $\overline{0}^{\mathcal{R}}$ over ψ



(b) Relation $\overline{1}^{\mathcal{R}}$ over ψ

Figure 2.3

Furthermore each of these relations can be used to define at least one mapping over ψ, for example mapping $\bar{0}^{\psi}$ of figure 1.14(a) was formed by associating just one of the $\bar{0}^{\mathcal{R}}$-successors of block(S_{10} S_{11}) with this block, and similarly mapping $\bar{1}^{\psi}$ of figure 1.14(b) was formed by associating just one of the two $\bar{1}^{\mathcal{R}}$-successors with block (S_{00} S_{01}). Defining $\overline{X}_{\psi} = \{\bar{0}^{\psi}, \bar{1}^{\psi}\}$ then gave the image system $T/\psi = \langle \psi \;\; \overline{X}_{\psi} \rangle$.

More generally, let ψ be a preserved cover of a semiautomaton $A = \langle S_A \;\; \overline{X}_A \rangle$ and associate with each input symbol $x \in X_A$ a relation $\bar{x}^{\mathcal{R}}$ over ψ, where
$\bar{x}^{\mathcal{R}} = \{ \langle B \;\; B' \rangle \mid B, B' \in \psi, \; (B)\bar{x}^{A} \neq \emptyset \;\; \& \;\; (B)\bar{x}^{A} \subseteq B' \}$.
From any relation $\bar{x}^{\mathcal{R}}$ at least one mapping $\bar{x}^{\psi}$ can be defined, by associating just one element of $[a]\bar{x}^{\mathcal{R}}$ with each element $a \in D[\bar{x}^{\mathcal{R}}]$, in fact this idea is closely related to the "axiom of choice" [Suppes (a)]. This produces a mapping $\bar{x}^{\psi}$ where $\bar{x}^{\psi} \subseteq \bar{x}^{\mathcal{R}}$ and $D[\bar{x}^{\psi}] = D[\bar{x}^{\mathcal{R}}]$, in which case mapping $\bar{x}^{\psi}$ will be said to be "derived" from relation $\bar{x}^{\mathcal{R}}$.

Definition

A mapping ρ' is derived from a relation ρ iff

$$\rho' \subseteq \rho \quad \text{and} \quad D[\rho'] = D[\rho].$$

Define $X_{\psi} = X_A$ and let $\overline{X}_{\psi}$ denote a set of mappings over ψ, so that if $x \in X_{\psi}$ then $\bar{x}^{\psi} \in \overline{X}_{\psi}$, where $\bar{x}^{\psi}$ is one of the mappings derived from relation $\bar{x}^{\mathcal{R}}$. Considering now the properties of unary algebra $\langle \psi \;\; \overline{X}_{\psi} \rangle$, assume $B \in \psi$ and

$x \in X_A$ where $(B)\bar{x}^A = \emptyset$. Then $B \notin D[\bar{x}^{\mathcal{R}}]$, and $D[\bar{x}^{\psi}] = D[\bar{x}^{\mathcal{R}}]$ so $B \notin D[\bar{x}^{\psi}]$, consequently $(\forall B)(\forall x)(B \in \psi,\ x \in X_A,\ (B)\bar{x}^A = \emptyset \Longrightarrow B \notin D[\bar{x}^{\psi}])$ in accordance with the definition of an image semiautomaton. Assuming now $B \in \psi$ and $x \in X_A$ where $(B)\bar{x}^A \neq \emptyset$, since ψ is a preserved cover there must be at least one cover block B^* where $(B)\bar{x}^A \subseteq B^*$, in which case $\langle B\ B^* \rangle \in \bar{x}^{\mathcal{R}}$. Furthermore mapping $\bar{x}^{\psi}$ over ψ is derived from relation $\bar{x}^{\mathcal{R}}$, that is $\bar{x}^{\psi} \subseteq \bar{x}^{\mathcal{R}}$ and $D[\bar{x}^{\psi}] = D[\bar{x}^{\mathcal{R}}]$ so $B \in D[\bar{x}^{\psi}]$, in which case $\langle B\ B' \rangle \in \bar{x}^{\psi}$ for some B', and $\bar{x}^{\psi} \subseteq \bar{x}^{\mathcal{R}}$ so $(B)\bar{x}^A \subseteq B'$. Therefore $(\forall B)(\forall x)(B \in \psi,\ x \in X_A,\ (B)\bar{x}^A \neq \emptyset \Longrightarrow (\exists B')(\langle B\ B' \rangle \in \bar{x}^{\psi} \;\&\; (B)\bar{x}^A \subseteq B'))$; confirming that $\langle \psi\ \bar{X}_{\psi} \rangle$ is an image semiautomaton.

This shows that an image semiautomaton can be formed from a given preserved cover by forming the relations $\bar{x}^{\mathcal{R}}$, and deriving a mapping $\bar{x}^{\psi}$ from each relation. However it is important to have established the image semiautomaton concept without direct reference to preserved covers, since image semiautomata can be formalised without progressing from a given preserved cover. An important example of this is encountered in weak-homomorphism "decomposition". Any weak homomorphism $A \xrightarrow{\theta} B$ defines an image semiautomaton of the "source" semiautomaton A, and can be decomposed into "intrinsic" and "extrinsic" components [Yoeli], just as a homomorphism defines a quotient algebra and is a product of an epimorphism and an isomorphism [Cohn]. Indeed weak-homomorphism decomposition also justifies the idea of a preserved cover as a family, since in decomposing a weak homomorphism the codomain must be used as an index set.

Clearly an image semiautomaton can be formed without progressing from a given preserved cover, but in all cases the canonical relation associated with an image semiautomaton is a weak homomorphism, just as the canonical relation H' associated with preserved cover ψ was shown to be a weak homomorphism of T to T/ψ .

Result [Yoeli]

The canonical relation from a semiautomaton A to an image semiautomaton G of A is a weak homomorphism.

If $G = \langle S_G \, \overline{X}_G \rangle$ is an image semiautomaton of $A = \langle S_A \, \overline{X}_A \rangle$ then $S_G = \pi_i$ where π_i is a S_A-cover, and the canonical relation from A to G is the canonical relation $\pi'_i = \left\{ \langle a \; P \rangle \mid a \in S_A, \; P \in \pi_i \;\; \& \;\; a \in P \right\}$ associated with cover π_i. Since π_i is a S_A-cover the relation π'_i has codomain $C[\pi'_i] = \pi_i = S_G$, so π'_i is a weak homomorphism of semiautomaton A "onto" G. That is, the "image semiautomaton" G is an image of A under the weak homomorphism π'_i.

2.4 Input tapes, and the associated semigroups

It is often necessary to consider the response of sequential circuits to sequences of input codes, and such a sequence can be represented as a mapping from the set $N = \{1,2,3 \; ... \}$ of nonzero integers to the set

$X_A = \{x_1, x_2, x_3, \ldots\}$ of the input symbols. For example the mapping $\{\langle 1\ x_1\rangle\ \langle 2\ x_2\rangle\ \langle 3\ x_1\rangle\}$ represents that the input code represented as x_1 is followed by the code represented as x_2, then the input code represented as x_1 recurs. For convenience the mapping is expressed as a "tape" $\langle x_1 x_2 x_1\rangle$,and in general a tape $t_p = \langle p_1 p_2 \ldots p_{u-1} p_u\rangle$ over an input set X_A represents input symbols from set X_A in an order, input symbol p_1 being "followed" by input symbol p_2, and so on. The set of all such tapes over the set X_A will be called the "tape set over X_A", and will be denoted $X_A{*}$.

<u>Definition</u>

For X_A any set, the <u>tape set over X_A</u> is the infinite set $X_A{*} = \lambda \cup X_A \cup X_A^2 \cup X_A^3 \cup X_A^4 \cup \ldots$

For example if $X_A = \{x_1, x_2, x_3, \ldots\}$ then x_1, $\langle x_1 x_2\rangle$, $\langle x_1 x_2 x_1\rangle$, $\langle x_1 x_2 x_1 x_3\rangle \in X_A{*}$ since $x_1 \in X_A$, $\langle x_1 x_2\rangle \in X_A{}^2$, $\langle x_1 x_2 x_1\rangle \in X_A^3$ and $\langle x_1 x_2 x_1 x_3\rangle \in X_A{}^4$.
A tape $t_p = \langle p_1 p_2 \ldots p_{u-1} p_u\rangle$ will be said to have "length" u, this being the integer associated with the last symbol of the tape. Each element from the input set X_A is considered to be a tape of unit length. Furthermore the above definition introduces the symbol λ denoting the "blank tape", and this tape is considered to have zero length. The blank tape can be considered to represent the absence of input codes, and the significance of this tape will be appreciated later.

Arbitrary tapes $t_p = \langle p_1 p_2 \ldots\ldots p_{u-1} p_u \rangle$ and $t_q = \langle q_1 q_2 \ldots\ldots q_{v-1} q_v \rangle$ can be combined in the obvious way to give the tape $\langle p_1 p_2 \ldots\ldots p_{u-1} p_u q_1 q_2 \ldots\ldots q_{v-1} q_v \rangle$, so that in effect one input sequence is directly followed by the other. This construction is formalised as the "concatenation operator", and the concatenation operator o is defined so that $t_p \circ t_q = \langle p_1 p_2 \ldots\ldots p_{u-1} p_u q_1 q_2 \ldots\ldots q_{v-1} q_v \rangle$ where t_p and t_q are the above arbitrary tapes, and for completeness $t_p \circ \lambda = \lambda \circ t_p = t_p$. The concatenation operator is "closed" over tape set $X_A{}^*$, that is $t_p, t_q \in X_A{}^*$ implies $t_p \circ t_q \in X_A{}^*$, confirmation of this being immediate for $t_p = \lambda$, or $t_q = \lambda$, or $t_p = t_q = \lambda$. Otherwise the tape t_p takes the form $t_p = \langle p_1 p_2 \ldots\ldots p_{u-1} p_u \rangle$, similarly $t_q = \langle q_1 q_2 \ldots\ldots q_{v-1} q_v \rangle$, in which case $t_p \in X_A{}^u$ and $t_q \in X_A{}^v$ so $t_p \circ t_q \in X_A{}^{u+v}$. Then $t_p \circ t_q \in X_A{}^*$, since $X_A{}^{u+v} \subseteq X_A{}^*$, and this confirms that the concatention operator is closed over $X_A{}^*$. Furthermore the operator is associative, that is $t_p \circ (t_q \circ t_r) = (t_p \circ t_q) \circ t_r$ for any tapes $t_p, t_q, t_r \in X_A{}^*$, so the system $\langle X_A{}^* \circ \rangle$ is a semigroup. In fact semigroup $\langle X_A{}^* \circ \rangle$ is the "free semigroup" generated by the set X_A, and is a "monoid" since, from the property $t_p \circ \lambda = \lambda \circ t_p = t_p$, the blank tape $\lambda \in X_A{}^*$ is an identity element.

Consider now a semiautomaton $A = \langle S_A\ \overline{X}_A \rangle$ where $X_A = \{x_1, x_2, x_3, \ldots\}$, and assume a_i, a_j, a_k, a_l are states from the state set S_A such that $\langle a_i\ a_j \rangle \in \overline{x}_1^A$, $\langle a_j\ a_k \rangle \in \overline{x}_2^A$ and $\langle a_k\ a_l \rangle \in \overline{x}_1^A$. Then the tape $\langle x_1 x_2 x_1 \rangle$ defines a series of assignments, starting with a_i and concluding with a_l, so that in effect the tape $\langle x_1 x_2 x_1 \rangle$ assigns state a_i to the "final successor" state a_l. To express the way a tape assigns final successors to semiautomaton states it is convenient to associate, with each tape $t_p \in X_A^*$, a mapping $\overline{t}_p^A$ over the semiautomaton state-set.

<u>Definition</u>

Let $A = \langle S_A\ \overline{X}_A \rangle$ be a semiautomaton and define $T_A = X_A^*$. Then each tape $t_p = \langle p_1 p_2 \ldots p_{u-1} p_u \rangle$ where $t_p \in T_A$ has an associated <u>final-state assignment</u> $\overline{t}_p^A : S_A \longrightarrow S_A$ where

$$\overline{t}_p^A = \overline{p}_1^A \cdot \overline{p}_2^A \cdot \overline{p}_3^A \cdot \ldots \overline{p_{u-1}}^A \cdot \overline{p}_u^A$$

and the blank tape $\lambda \in T_A$ has the associated mapping $\overline{\lambda}^A : S_A \longrightarrow S_A$ where $\overline{\lambda}^A = \Delta[S_A]$.

The set of all the mappings $\overline{t}_p^A$ where $t_p \in T_A$ will be denoted $\overline{T}_A$.

For a semiautomaton $A = \langle S_A\ \overline{X}_A \rangle$ representing the state transitions of a sequential circuit, $\langle a_i\ a_l \rangle \in \overline{t}^A$ expresses that the sequence of input codes represented by the tape t, when applied to the circuit in the state represented as a_i, will leave the circuit in the state represented as a_l.

For convenience the dot denoting the usual composition of mappings will sometimes be omitted, for example the mapping associated with the tape $\langle x_1 x_2 x_1 \rangle$ might be written $\bar{x}_1^A\ \bar{x}_2^A\ \bar{x}_1^A$.

To any tape $t \in T_A$ corresponds a mapping $\bar{t}^A$ over S_A where $\bar{t}^A \in \bar{T}_A$, so the set $\bar{T}_A$ is indexed by the tape set T_A, just as the transition set $\bar{X}_A$ of the semiautomaton is indexed by the input set X_A. Furthermore the tape set T_A is an infinite set, but if S_A is finite the set $\bar{T}_A$ must also be finite. In fact if $|S_A| = n$ then $\bar{T}_A$ can have at most n^n elements, and in general $\bar{T}_A$ will have less than n^n elements since $\bar{T}_A$ contains only the mappings over S_A constructed ~~from~~ the mappings forming $\bar{X}_A$. It is interesting also to consider the composition of the mappings within $\bar{T}_A$. The mappings $\bar{t}_p^A, \bar{t}_q^A \in \bar{T}_A$ can be used to form a mapping $\bar{t}_p^A \cdot \bar{t}_q^A$ over S_A, and in fact $\bar{t}_p^A \cdot \bar{t}_q^A = \overline{t_p \circ t_q}^A$. To appreciate this assume as before $t_p = \langle p_1 \ldots\ldots p_u \rangle$ and $t_q = \langle q_1 \ldots\ldots q_v \rangle$, so that $t_p \circ t_q = \langle p_1 \ldots\ldots p_u q_1 \ldots\ldots q_v \rangle$. Then

$$\overline{t_p \circ t_q}^A = \bar{p}_1^A \cdot \ldots\ldots \cdot \bar{p}_u^A \cdot \bar{q}_1^A \ldots\ldots \cdot \bar{q}_v^A$$

$$= (\bar{p}_1^A \cdot \ldots\ldots \cdot \bar{p}_u^A) \cdot (\bar{q}_1^A \ldots\ldots \cdot \bar{q}_v^A)$$

so $\quad \overline{t_p \circ t_q}^A = \bar{t}_p^A \cdot \bar{t}_q^A$

Furthermore $\bar{t}_p^A, \bar{t}_q^A \in \bar{T}_A$ implies $\bar{t}_p^A \cdot \bar{t}_q^A \in \bar{T}_A$, since $\bar{t}_p^A, \bar{t}_q^A \in \bar{T}_A$ implies $t_p, t_q \in T_A$, in which case

$t_p \circ t_q \in T_A$ so $\overline{t_p \circ t_q}^A \in \overline{T}_A$, that is $\overline{t}_p^A \circ \overline{t}_q^A \in \overline{T}_A$. Consequently the composition of mappings is closed over $\overline{T}_A$, and the composition is also associative, that is $\overline{t}_p^A \bullet (\overline{t}_q^A \bullet \overline{t}_r^A) = (\overline{t}_p^A \bullet \overline{t}_q^A) \bullet \overline{t}_r^A$ for any $\overline{t}_p^A, \overline{t}_q^A, \overline{t}_r^A \in \overline{T}_A$, since composition is always associative [Suppes (a)]. Therefore $\langle \overline{T}_A \circ \rangle$ is a semigroup, indeed $\langle \overline{T}_A \bullet \rangle$ is a monoid since, from the property $\overline{\lambda}^A = \Delta[S_A]$, $\overline{\lambda}^A \in \overline{T}_A$ is an identity element.

Consequently the semiautomaton $A = \langle S_A \; \overline{X}_A \rangle$ has an associated "input semigroup" $\langle T_A \; o \rangle$, where $T_A = X_A^*$, and a transition semigroup $\langle \overline{T}_A \; \circ \rangle$. The relationship between semiautomata and semigroups has inspired extensive recent research [Arbib], however interest is restricted at present to the important identity $\overline{t_p \circ t_q}^A = \overline{t}_p^A \circ \overline{t}_q^A$. Clearly the indexing surjection, relating each tape t_p of the infinite set $T_A = X_A^*$ to the corresponding mapping $\overline{t}_p^A$ from the finite set $\overline{T}_A$, forms the commutative graph of figure 2.4, so that intuitively $\overline{t_p \circ t_q}^A = \overline{t}_p^A \circ \overline{t}_q^A$ expresses a homomorphism. In fact the indexing surjection from T_A to $\overline{T}_A$ is a homomorphism of the semigroup $\langle T_A \; o \rangle$ onto the semigroup $\langle \overline{T}_A \; \circ \rangle$, that is the indexing surjection is an epimorphism, and $\langle \overline{T}_A \; \circ \rangle$ is a homomorphic image of $\langle T_A \; o \rangle$.

$$\begin{tikzcd}
\underbrace{t_p \in T_A,\ t_q \in T_A} \arrow[d, "\text{concatenation}"] \arrow[r, bend left] & \underbrace{\overline{t_p}^A \in \overline{T}_A,\ \overline{t_q}^A \in \overline{T}_A} \arrow[d, "\text{composition}"] \\
\begin{matrix} t_p \circ t_q \\ \in T_A \end{matrix} \arrow[r] & \begin{matrix} \overline{t_p}^A \bullet \overline{t_q}^A \\ \in \overline{T}_A \end{matrix}
\end{tikzcd}$$

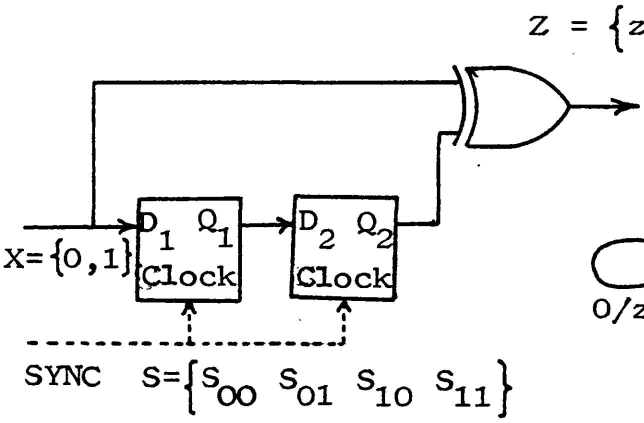
Figure 2.4 $\underline{\overline{t_p \circ t_q}^A = \overline{t_p}^A \bullet \overline{t_q}^A}$

2.5 Automata

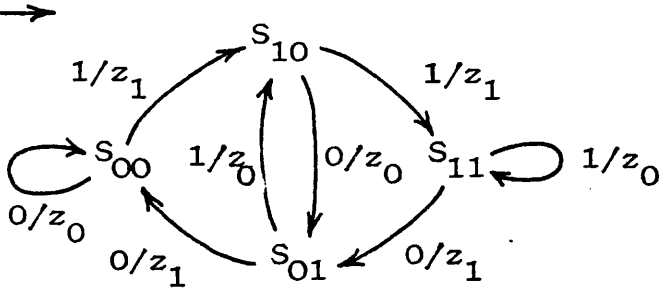
A sequential circuit is of Moore type [Moore] if the output code depends only on the existing state-code. For such a circuit the production of output codes can be expressed as a mapping $\omega : S \longrightarrow Z$, and the overall action of the circuit can be expressed as a quintuple $\langle S\ X\ Z\ \overline{X}\ \omega \rangle$, where state-set S, input set X and output set Z represent the codes associated with the circuit.

The representation of sequential circuits of Mealy type [Mealy], where the output code is determined by the existing state-code and the existing input code, is less straightforward. This dependence is usually expressed as a mapping $\omega : S \times X \longrightarrow Z$, however the Cartesian product can be avoided by associating a mapping from S to Z with each input symbol. More specifically let $S = \{s_1, s_2, s_3, \ldots\}$ represent the state codes associated with the circuit, let $X = \{x_1, x_2, x_3, \ldots\}$ represent the input codes and let $Z = \{z_1, z_2, z_3, \ldots\}$ represent the output codes. Then input symbol x_1 will associate some output symbol $z_i \in Z$ with state s_1, in the sense that the input code represented as x_1 produces the output code represented as z_i when applied to the circuit in the state represented as s_1. Similarly input x_1 will associate some output symbol z_j with state symbol s_2, will associate some output symbol z_k with state symbol s_3, and so on, and these associations define a set $\tilde{x}_1 = \{\langle s_1\ z_i\rangle\langle s_2\ z_j\rangle\langle s_3\ z_k\rangle \ldots\}$. Then $\tilde{x}_1$ is a mapping from state set S to output set Z, and $\langle s\ z\rangle \in \tilde{x}_1$ expresses that input symbol x_1 associates output symbol z with the state symbol s. Similarly the input symbol x_2 defines a mapping $\tilde{x}_2 : S \longrightarrow Z$, input symbol x_3 defines a mapping $\tilde{x}_3 : S \longrightarrow Z$, and so on, giving a set $\tilde{X} = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots\}$ of mappings from the state set S to the output set Z. Furthermore set $\tilde{X}$ is indexed by set X, so that to any input symbol $x_i \in X$ corresponds a particular mapping $\tilde{x}_i : S \longrightarrow Z$, and if $|S| = m$ and $|Z| = n$ then $\tilde{X}$ has at most n^m elements.

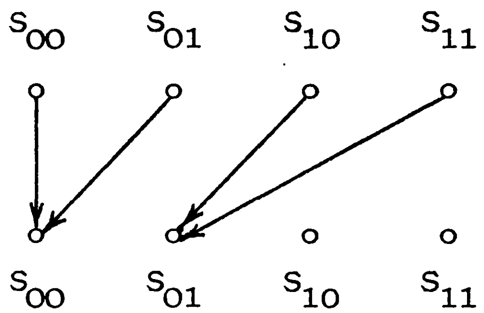
To illustrate these ideas, consider the simple Mealy circuit of figure 2.5(a). The circuit incorporates two delay bistables in the form of a two-stage shift-register, and output codes are generated by an exclusive-or gate.



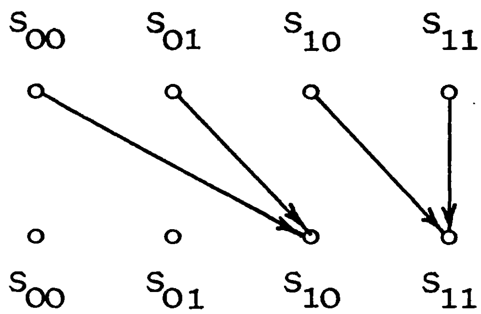
(a) Sequential circuit of Mealy type



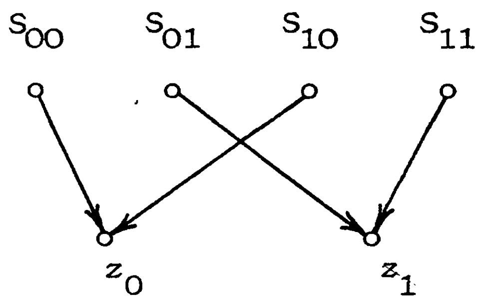
(b) Graphical representation



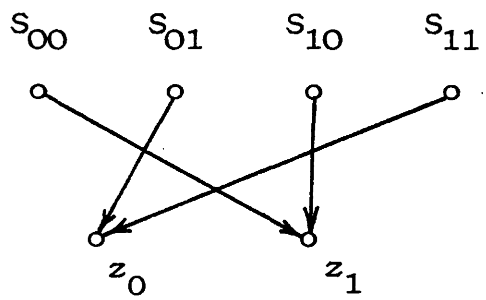
(c) Mapping $\bar{0} : S \longrightarrow S$



(d) Mapping $\bar{1} : S \longrightarrow S$



(e) Mapping $\tilde{0} : S \longrightarrow Z$



(f) Mapping $\tilde{1} : S \longrightarrow Z$

Figure 2.5

Then the set $S = \{S_{00}\ S_{01}\ S_{10}\ S_{11}\}$ of states S_{Q1Q2}, the input set $X = \{0,1\}$ and the output set $Z = \{z_0, z_1\}$ represent the codes associated with the circuit, and the conventional graphical representation for the circuit is shown in figure 2.5(b). The state transitions of the two-stage shift register were previously expressed as the mappings $\overline{0}$ and $\overline{1}$ of figure 1.10, and for convenience these mappings are redrawn as the figures 2.5(c) and 2.5(d). Considering now the outputs produced by the circuit, an arc from state S_{00} on figure 2.5(b) is labelled $0/z_0$ and this expresses that input 0 associates output z_0 with state S_{00}. Similarly input 0 associates output z_1 with state S_{01}, associates output z_0 with state S_{10} and associates output z_1 with state S_{11}, giving the mapping $\tilde{0}$ from S to Z where

$\tilde{0} = \{\langle S_{00}\ z_0\rangle\ \langle S_{01}\ z_1\rangle\ \langle S_{10}\ z_0\rangle\ \langle S_{11}\ z_1\rangle\}$, as shown in figure 2.5 (e). Consideration of input 1 defines the associated mapping $\tilde{1}$ from S to Z where

$\tilde{1} = \{\langle S_{00}\ z_1\rangle\ \langle S_{01}\ z_0\rangle\ \langle S_{10}\ z_1\rangle\ \langle S_{11}\ z_0\rangle\}$, as in figure 2.5(f). Then the circuit can be represented as the "Mealy automaton" $\langle S\ X\ Z\ \overline{X}\ \tilde{X}\rangle$, where $\overline{X} = \{\overline{0}, \overline{1}\}$ and $\tilde{X} = \{\tilde{0}, \tilde{1}\}$.

Similarly a circuit of Moore type would be represented as a "Moore automaton", the formal definitions being given overleaf. It is evident that automata representing sequential circuits will always be finite, and will always be complete, so it is important to justify interest in "partial" automata. In the subsequent study of sequential circuit synthesis the design objective will always be expressed as an "objective automaton", and it will usually be possible to form an objective automaton without associating an output with every state, and without associating a successor with every state for every input. In fact such partial automata will dominate the subsequent studies and the term "automaton" will always relate to partial automata, "complete" automata being explicitely qualified.

Definition

A Moore automaton is a system $\hat{A} = \langle S_A\ X_A\ Z_A\ \overline{X}_A\ \omega_A \rangle$ where S_A, X_A and Z_A are nonvoid sets, $A = \langle S_A\ \overline{X}_A \rangle$ is a X_A-semiautomaton over S_A and ω_A is a mapping from S_A to Z_A.

Set S_A is the state set of the automaton, X_A is the input set and Z_A is the output set. The automaton is finite iff S_A, X_A and Z_A are finite sets. Set $\overline{X}_A$ is the transition set of the automaton, and ω_A is the output mapping.

Automaton $\hat{A}$ is transition complete iff $A = \langle S_A\ \overline{X}_A \rangle$ is a complete semiautomaton, and is output complete iff $D[\omega_A] = S_A$. The automaton is complete iff both transition complete and output complete, otherwise the automaton is partial.

Definition

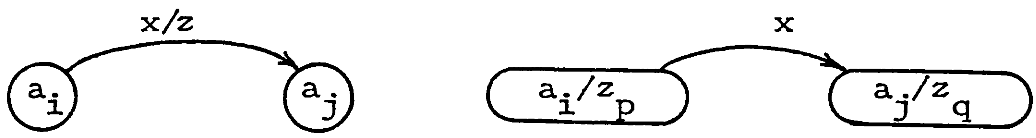
A Mealy automaton is a system $\hat{A} = \langle S_A\ X_A\ Z_A\ \overline{X}_A\ \tilde{X}_A \rangle$ where S_A, X_A and Z_A are nonvoid sets, $A = \langle S_A\ \overline{X}_A \rangle$ is a X_A-semiautomaton over S_A, and $\tilde{X}_A$ is a X_A-indexed set of mappings $\tilde{x}^A$ from S_A to Z_A.

Set S_A is the state set of the automaton, X_A is the input set and Z_A is the output set. The automaton is finite iff S_A, X_A and Z_A are finite sets. Set $\overline{X}_A$ is the transition set of the automaton and set $\tilde{X}_A$ is the response set.

Automaton $\hat{A}$ is transition complete iff $A = \langle S_A\ \overline{X}_A \rangle$ is a complete semiautomaton, and is output complete iff $(\forall x)(x \in X_A \Longrightarrow D[\tilde{x}^A] = S_A)$. The automaton is complete iff both transition and output complete, otherwise the automaton is partial.

The properties of partial automata extend, of course, to complete automata, which can be regarded as "refined" partial automata.

It is also important to formalise the conventional graphical representations for Moore and Mealy automata. The representation for the Mealy quintuple was encountered in figure 2.5(b), and in general the automaton states are represented as graph nodes with associations $\langle a_i \; a_j \rangle \in \bar{x}^A$, $\langle a_i \; z \rangle \in \tilde{x}^A$ involving state a_i, state a_j, input symbol x and output symbol z represented as in figure 2.6(a). In representing a Moore quintuple an output symbol is associated with each node, for example figure 2.6(b) expresses $\langle a_i \; z_p \rangle \in \omega_A$, $\langle a_j \; z_q \rangle \in \omega_A$ and $\langle a_i \; a_j \rangle \in \bar{x}^A$.



(a) $\langle a_i \; a_j \rangle \in \bar{x}^A$, $\langle a_i \; z \rangle \in \tilde{x}^A$

(b) $\langle a_i z_p \rangle \in \omega_A$, $\langle a_j \; z_q \rangle \in \omega_A$ and $\langle a_i \; a_j \rangle \in \bar{x}^A$

Figure 2.6

The response of automaton $\langle S \; X \; Z \; \bar{X} \; \tilde{X} \rangle$ to a tape such as $t = \langle 1011 \rangle$ is evident from figure 2.5(b), for example state S_{00} has $\bar{1}$-successor S_{10}, state S_{10} has $\bar{0}$-successor S_{01}, state S_{01} has $\bar{1}$-successor S_{10} and state S_{10} has $\bar{1}$-successor S_{11}, so tape $t = \langle 1011 \rangle$ defines a series of state transitions starting with S_{00} and ending with S_{11}. This is illustrated in figure 2.7, it being assumed that state transitions and changes of input occur instantaneously, in accordance with a synchronisation signal.

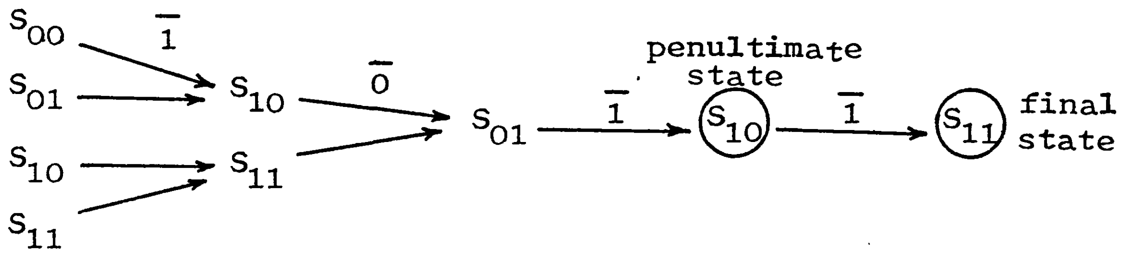
	SYNC	SYNC	SYNC	SYNC
1 (initial input)	0	1	1	
S_{00} (initial state)	S_{10}	S_{01}	S_{10} (penultimate state)	S_{11} (final state)
z_1 (initial output)	z_0	z_0	z_1 (final output)	

Figure 2.7

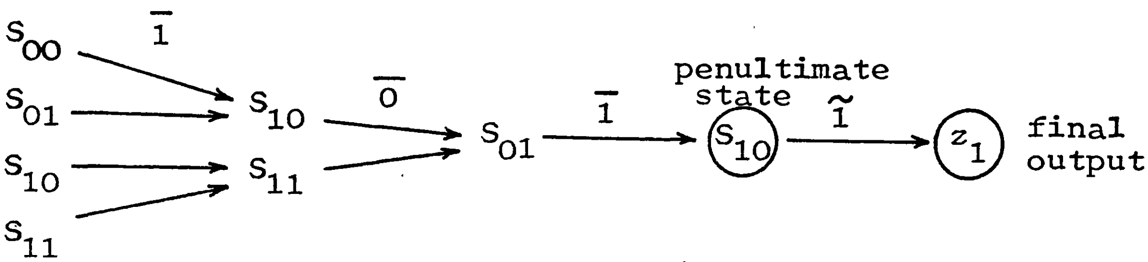
For the first synchronisation signal the state is S_{00} and the input is 1, so the subsequent state is S_{10}, furthermore a new input is established so for the second synchronisation signal the state is S_{10} and the input is 0. The figure also shows the corresponding sequence of output symbols, for example input 1 associates output z_1 with state S_{00}, input 0 associates output z_0 with state S_{10}, and so on, so it is clear that the circuit will produce an output for each input code, and that the final output coincides with the penultimate state.

The response of the circuit to the tape $t = \langle 1011 \rangle$ is expressed more completely in figure 2.8, for example the figure confirms that this tape associates final state S_{11}

with state S_{00} since $\langle S_{00}\ S_{11}\rangle \in \bar{t}$ where $\bar{t} = \bar{1}\,\bar{0}\,\bar{1}\,\bar{1}$, and shows that the tape associates penultimate state S_{10} with S_{00} since $\langle S_{00}\ S_{10}\rangle \in \bar{1}\,\bar{0}\,\bar{1}$. Furthermore figure 2.8(b) shows that the tape $t = \langle 1\,0\,1\,1\rangle$ associates final output z_1 with state S_{00} since $\tilde{1}$ associates this output with the penultimate state S_{10}, that is since $\langle S_{00}\ z_1\rangle \in \bar{1}\,\bar{0}\,\bar{1}\,\tilde{1}$. To express the assignment of final outputs to the automaton states, it is useful to associate a mapping $\tilde{t}_p$ from S to Z with each tape t_p. For a Mealy automaton the mapping $\tilde{t}_p$ to be associated with a tape $t_p = \langle p_1 p_2 \ \cdots\ p_{u-1} p_u\rangle$ is given by $\tilde{t}_p = \bar{p}_1\bar{p}_2 \ \cdots\ \overline{p_{u-1}}\tilde{p}_u$, since the final output is that associated with the penultimate state by the final input. For a Moore automaton, the mapping to be associated with t_p is $\tilde{t}_p = \bar{t}_p\,\omega$ since the final output is the output associated with the final state.



(a) Mapping $\bar{1}\,\bar{0}\,\bar{1}\,\bar{1}$ over S



(b) Mapping $\bar{1}\,\bar{0}\,\bar{1}\,\tilde{1}$ from S to Z

Figure 2.8 Response to tape $t = \langle 1\,0\,1\,1\rangle$

Definition

Let automaton $\hat{A}$ have tape set T_A, state set S_A and output set Z_A. Then any tape $t_p = \langle p_1 p_2 \ldots p_{u-1} p_u \rangle$ from T_A has an associated <u>final-output mapping</u> $\tilde{t}_p^A : S_A \longrightarrow Z_A$, where

$$\tilde{t}_p^A = \overline{t}_p^A \omega_A \quad \text{for } \hat{A} \text{ a Moore automaton,}$$

$$\tilde{t}_p^A = \overline{p_1}^A \, \overline{p_2}^A \ldots \overline{p_{u-1}}^A \, \tilde{p}_u^A \quad \text{for } \hat{A} \text{ a Mealy automaton.}$$

Furthermore $\tilde{\lambda}^A = \omega_A$ for $\hat{A}$ a Moore automaton, otherwise $\tilde{\lambda}^A = \emptyset$.

The set of all the mappings $\tilde{t}_p^A$ where $t_p \in T_A$ will be denoted $\tilde{T}_A$.

Clearly $\widetilde{t_p \circ t_q}^A = \overline{t}_p^A \cdot \tilde{t}_q^A$, since tapes $t_p = \langle p_1 \ldots p_u \rangle$ and $t_q = \langle q_1 \ldots q_{v-1} q_v \rangle$ form the tape $t_p \circ t_q = \langle p_1 \ldots p_u q_1 \ldots q_{v-1} q_v \rangle$, and then for a Mealy automaton

$$\begin{aligned}
\widetilde{t_p \circ t_q}^A &= \overline{p_1}^A \ldots \overline{p_u}^A \cdot \overline{q_1}^A \cdot \ldots \overline{q_{v-1}}^A \cdot \tilde{q}_v^A \\
&= (\overline{p_1}^A \ldots \overline{p_u}^A) \cdot (\overline{q_1}^A \ldots \overline{q_{v-1}}^A \, \tilde{q}_v^A) \\
&= \overline{t}_p^A \cdot \tilde{t}_q^A ,
\end{aligned}$$

and in the case of a Moore automaton

$$\begin{aligned}
\widetilde{t_p \circ t_q}^A &= \overline{t_p \circ t_q}^A \cdot \omega_A \\
&= \overline{t}_p^A \cdot \overline{t}_q^A \cdot \omega_A \\
&= \overline{t}_p^A \cdot \tilde{t}_q^A
\end{aligned}$$

The mappings $\overline{t_p}^A$ and $\tilde{t}_p^A$ associated with a given tape t_p express that the tape can be considered to associate a "final successor" and a "final output" with each automaton state. There is, however, a converse view, since each automaton state can be considered to associate a terminating state and a final output with each tape. Specifically, a state a_i will associate with the tape t_p the terminating state a_j where $\langle a_i \; a_j \rangle \in \overline{t_p}$, and will associate with tape t_p the final output z where $\langle a_i \; z \rangle \in \tilde{t}_p^A$. To express that each automaton state can associate a terminating state and a final output with each tape, it is convenient to associate a mapping $\Sigma a_i : X_A^* \longrightarrow S_A$ and a mapping $\Gamma a_i : X_A^* \longrightarrow Z_A$ with each automaton state a_i.

<u>Definition</u>

Let automaton $\hat{A}$ (either Mealy or Moore) have tape set T_A, state set S_A and output set Z_A.

Then to each state $a_i \in S_A$ corresponds <u>state-mappings</u> $\Sigma a_i : T_A \longrightarrow S_A$ and $\Gamma a_i : T_A \longrightarrow Z_A$, where

$$\langle t \; a_j \rangle \in \Sigma a_i \quad \text{iff} \quad \langle a_i \; a_j \rangle \in \overline{t}^A$$

$$\text{and} \quad \langle t \; z \rangle \in \Gamma a_i \quad \text{iff} \quad \langle a_i \; z \rangle \in \tilde{t}^A$$

The state-mappings Σa_i are particularly useful in semiautomaton analysis, for example a semiautomaton $A = \langle S_A \; \overline{X}_A \rangle$ is defined to be "a_i-connected" if the mapping Σa_i has codomain $C[\Sigma a_i] = S_A$, and the semiautomaton is "strongly connected" if connected to each state, that is if each of the mappings Σa_i for $a_i \in S_A$ has codomain $C[\Sigma a_i] = S_A$. In contrast, the mapping $\Gamma a_i : T_A \longrightarrow Z_A$ formalises the idea of a sequential circuit as a transducer of input tapes to output codes [Rabin & Scott], since $\langle t\ z \rangle \in \Gamma a_i$ expresses that the sequence represented as t can be applied to the circuit in the state represented as a_i, and the response will be the output code represented as z. The state-mappings Γa_i associated with an automaton are of fundamental importance, for example the relation of state-equivalence and the relation of state-compatibility can be defined directly.

<u>Definition</u>

Let $\hat{A}$ be an automaton with state set S_A.

The relation of <u>state-equivalence</u> over S_A is the relation denoted $\equiv$ where, for any states $a_i, a_j \in S_A$,

$$a_i \equiv a_j \quad \text{iff} \quad \Gamma a_i = \Gamma a_j$$

<u>Definition</u>

Let $\hat{A}$ be an automaton with state set S_A and input set X_A.

The relation of <u>state-compatibility</u> over S_A is the relation denoted $\approx$ where, for any states $a_i, a_j \in S_A$,

$a_i \approx a_j$ iff

$$(\forall t)(\forall z_i)(\forall z_j)(t \in X_A^*,\ \langle t\ z_i \rangle \in \Gamma a_i,\ \langle t\ z_j \rangle \in \Gamma a_j \Rightarrow z_i = z_j)$$

Each of the above relations is preserved, of course, under the transition mappings $\bar{x}^A$ where $x \in X_A$, indeed

$X_A \subseteq X_A^*$, and the relations are preserved under any mapping $\overline{t_p}^A$ where $t_p \in X_A^*$. For example assume $a_i \approx a_j$, $\langle a_i\ a'_i \rangle \in \overline{t_p}^A$, and $\langle a_j\ a'_j \rangle \in \overline{t_p}^A$ where $t_p \in X_A^*$. To confirm $a'_i \approx a'_j$, showing that state-compatibility is preserved under $\overline{t_p}^A$, assume $\langle t\ z_i \rangle \in \Gamma a'_i$ and $\langle t\ z_j \rangle \in \Gamma a'_j$. Then $\langle a'_i\ z_i \rangle \in \tilde{t}^A$, since $\langle t\ z_i \rangle \in \Gamma a'_i$ iff $\langle a'_i\ z_i \rangle \in \tilde{t}^A$, furthermore $\langle a_i\ a'_i \rangle \in \overline{t_p}^A$ so $\langle a_i\ z_i \rangle \in \overline{t_p}^A \tilde{t}^A$, that is $\langle a_i\ z_i \rangle \in \widetilde{t_p \circ t}^A$. Hence $\langle t_p \circ t, z_i \rangle \in \Gamma a_i$, and similarly $\langle t_p \circ t, z_j \rangle \in \Gamma a_j$, and $a_i \approx a_j$ so $z_i = z_j$. Therefore $\langle t\ z_i \rangle \in \Gamma a'_i$, $\langle t\ z_j \rangle \in \Gamma a'_j$ implies $z_i = z_j$, in which case $a'_i \approx a'_j$.

This confirms that state-compatibility is preserved under any final-state assignment $\overline{t_p}^A$ where $t_p \in X_A^*$, and $X_A \subseteq X_A^*$ so the relation is preserved under any transition mapping $\bar{x}^A$ from $\overline{X}_A$. Clearly state-compatibility, and similarly state-equivalence, is preserved within the unary algebra $\langle S_A\ \overline{X_A^*} \rangle$, in fact any relation is preserved within $\langle S_A\ \overline{X_A^*} \rangle$ iff preserved within $\langle S_A\ \overline{X}_A \rangle$. It also follows that the cover associated with the state-compatibility relation is preserved under the mappings $\overline{t_p}^A$ from $\overline{X_A^*}$, and is therefore a preserved cover of $\langle S_A\ \overline{X_A^*} \rangle$, and similarly state-equivalence defines a preserved partition.

It is interesting to observe that the state mappings Γa_1, Γa_2, Γa_3, ... associated with the respective states

$a_1, a_2, a_3, \ldots$ of an automaton are closely interrelated, and to appreciate this let X_A be the input set of an automaton $\hat{A}$. Then to each input symbol $x \in X_A$ corresponds a mapping $L_x : X_A^* \longrightarrow X_A^*$ of "left concatenation by x", where $\langle t, x \circ t \rangle \in L_x$ for any tape $t \in X_A^*$, that is L_x assigns each tape $t \in X_A^*$ to the tape $x \circ t \in X_A^*$, this being the tape produced by prefixing or "left concatenating" tape t using input symbol x. Now let S_A be the state set of automaton $\hat{A}$, assume $a_i, a_i' \in S_A$ where $\langle a_i\ a_i' \rangle \in \bar{x}^A$, and assume $\langle t\ z \rangle \in \Gamma a_i'$ for some tape t and some output z. Then a_i' is the x-successor of state a_i, and a_i' associates output symbol z with tape t, as figure 2.9 illustrates. Then state a_i must associate output z with the tape $x \circ t$, that is $[x \circ t]\ \Gamma a_i = [t]\ \Gamma a_i' = \{z\}$, and
$[x \circ t]\ \Gamma a_i = [t] L_x \Gamma a_i$ so $\Gamma a_i' = L_x \Gamma a_i$

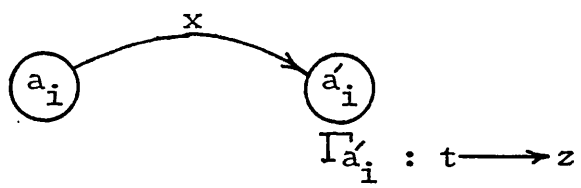


Figure 2.9 $\Gamma a_i' = L_x \Gamma a_i$

Consequently each automaton state a_i can be considered to represent a state mapping Γa_i, and each state transition can be considered to represent an identity of the form $\Gamma a_i' = L_x \Gamma a_i$. This suggests that the design objective, in sequential circuit synthesis, can be expressed in automaton form by expressing the objective as a mapping $\ulcorner a_1 : X^* \longrightarrow Z$, and finding for each $x \in X$ a finite system of identities $\ulcorner a_2 = L_x \ulcorner a_1$, $\ulcorner a_3 = L_x \ulcorner a_2$, ... These mappings $\ulcorner a_1$, $\ulcorner a_2$, $\ulcorner a_3$, ... then become the states of the objective automaton, and the identities define the state transitions. However, this does not provide a practical approach for forming an objective automaton. Consideration of the work of Nerode, and that of Kleene, is much more lucrative.

2.6 Automaton Analysis

An arbitrary, complete Moore automaton $\hat{A} = \langle S_A\ X_A\ Z_A\ \overline{X}_A\ \omega_A \rangle$ can be analysed with respect to any particular state a_i from S_A, by considering the mappings $\Sigma a_i : X_A^* \longrightarrow S_A$ and $\Gamma a_i : X_A^* \longrightarrow Z_A$ associated with this "reference" state. By definition

$$\Sigma a_i = \left\{ \langle t\ a \rangle \middle| t \in X_A^*,\ \ a \in S_A\ \ \&\ \ \langle a_i\ a \rangle \in \overline{t}^A \right\},$$

and the mapping can be visualised as shown in figure 2.10.

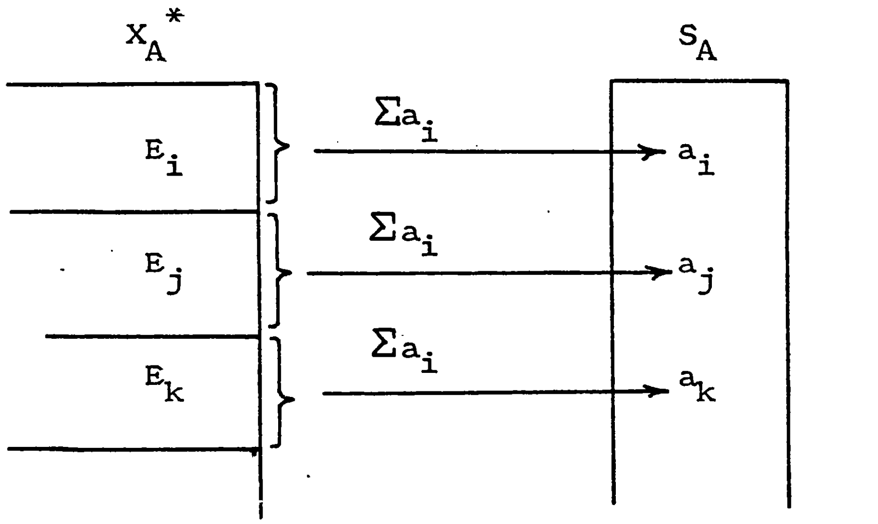


Figure 2.10 Mapping $\Sigma a_i : X_A^* \longrightarrow S_A$

Each state "accessible" from the reference state a_i, that is each state from the codomain $C[\Sigma a_i]$, defines a subset of X_A^* consisting of the tapes assigned to this state by mapping Σa_i. For example the figure shows that the state a_j defines a subset E_j of X_A^*, where E_j consists of all the tapes assigned to a_j by Σa_i, in which case $E_j = [a_j]\, \Sigma a_i^{-1}$ or
$E_j = \left\{ t \middle| t \in X_A^*\ \ \&\ \ \langle a_i\ a_j \rangle \in \overline{t}^A \right\}$. In particular the

subset $E_i \subseteq X_A^*$ associated with the reference state a_i consists of all the tapes $t \in X_A^*$ where $\langle a_i\ a_i \rangle \in \overline{t}^A$, and $\lambda \in E_i$ since $\overline{\lambda}^A = \Delta[S_A]$.

Since automaton $\hat{A}$ is complete each tape from X_A^* must assign reference state a_i to some specific successor-state, so the subsets defined by the states will form a partition of X_A^*. There will be an equivalence class for each state accessible from state a_i, so that for S_A finite the infinite set X_A^* will be divided into a finite number of equivalence classes. More formally the kernel $\ker(\Sigma a_i) = \Sigma a_i (\Sigma a_i)^{-1}$ of mapping Σa_i is an equivalence over the domain $D[\Sigma a_i] = X_A^*$, and has an equivalence class for each element of codomain $C[\Sigma a_i]$, so that if $A = \langle S_A\ \overline{X}_A \rangle$ is a_i-connected there will be an equivalence class associated with each state. For convenience the equivalence $\ker(\Sigma a_i)$ over X_A^* will be denoted R_i, the associated partition of X_A^* being denoted X_A^*/R_i.

A significant property of partition X_A^*/R_i can be expressed by extending the concatenation symbology, so that if E_p and E_q are given sets of tapes $E_p \circ E_q$ will denote the set of all the tapes $t_p \circ t_q$ where $t_p \in E_p$ and $t_q \in E_q$. For additional convenience no distinction will be made between a tape t and the related singleton $\{t\}$, for example $E_j \circ \{t\}$ will be written $E_j \circ t$ and $E_j \circ \{x\}$ will be written $E_j \circ x$. Having established this symbology, let a_j be some state of the above automaton $\hat{A}$, let E_j be the associated equivalence class of X_A^*/R_i and assume $x \in X_A$. Since automaton $\hat{A}$ is complete every state will have a $\bar{x}^A$-successor, and in particular there must be a $\bar{x}^A$-successor a'_j for a_j. Furthermore any tape from E_j will assign a_i to final-successor state a_j, so any tape from $E_j \circ x$ will assign a_i to the $\bar{x}^A$-successor of a_j, that is any tape from $E_j \circ x$ assigns a_i to a'_j. Consequently $E_j \circ x \subseteq E'_j$, where E'_j is the equivalence class associated with a'_j.

The preceeding shows that if $A = \langle S_A \; \bar{X}_A \rangle$ is a complete semiautomaton, E_j is an equivalence class of the partition X_A^*/R_i, and $x \in X_A$, then there will be an equivalence class E'_j where $E_j \circ x \subseteq E'_j$, specifically E'_j is the equivalence class associated with the $\bar{x}^A$-successor of a_j. Alternatively, this can be expressed as a property of the equivalence $R_i = \ker(\Sigma a_i)$ associated with partition X_A^*/R_i, in fact relation R_i is "right-invariant under concatenation" [Rabin & Scott].

<u>Definition</u>

An equivalence R over X_A^* is <u>right-invariant under concatenation</u> iff

$$(\forall x)(\forall t_p)(\forall t_q)(x \in X_A,\ t_p \equiv t_q \ (R) \Longrightarrow t_p \circ x \equiv t_q \circ x \ (R))$$

Here the symbolism $t_p \equiv t_q$ (R) means that the tapes t_p and t_q belong to a common equivalence class of the relation R, in particular $t_p \equiv t_q$ (R_i) expresses that the tapes t_p and t_q associate a common final-successor state with reference state a_i. To verify that relation R_i is right-invariant assume $x \in X_A$, and assume $t_p \equiv t_q\ (R_i)$. Then $\langle a_i\ a_j \rangle \in \overline{t_p}^A$ and $\langle a_i\ a_j \rangle \in \overline{t_q}^A$ for some state a_j, and automaton $\hat{A}$ is complete so a_j must have some $\bar{x}^A$-successor, that is $\langle a_j\ a_j' \rangle \in \bar{x}^A$ for some state a_j'. Hence $\langle a_i\ a_j \rangle \in \overline{t_p}^A$ and $\langle a_j\ a_j' \rangle \in \bar{x}^A$, giving $\langle a_i\ a_j' \rangle \in \overline{t_p}^A\ \bar{x}^A$, that is $\langle a_i\ a_j' \rangle \in \overline{t_p \circ x}^A$, and similarly $\langle a_i\ a_j' \rangle \in \overline{t_q \circ x}^A$, in which case $t_p \circ x \equiv t_q \circ x\ (R_i)$.

This confirms that relation R_i is right-invariant, so that in effect the equivalence of tapes is "preserved" under concatenation, and this can be investigated further by expressing right-concatenation as a set of mappings over X_A^*. For example right-concatenation using an input symbol x_1 from X_A will convert the tapes $t_p, t_q, t_r, \ldots \in X_A^*$ to the tapes $t_p \circ x_1,\ t_q \circ x_1,\ t_r \circ x_1, \ldots \in X_A^*$, and this can be expressed as a mapping $\mathring{x}_1$ over X_A^* where $\langle t, t \circ x_1 \rangle \in \mathring{x}_1$ for any tape $t \in X_A^*$. Similarly a mapping $\mathring{x}_2$ over X_A^* expresses right-concatenation using the input symbol x_2 from X_A, a mapping $\mathring{x}_3$ over X_A^* expresses right-concatenation using input symbol x_3, and so on, to give a finite set $\mathring{X}_A = \{\mathring{x}_1, \mathring{x}_2, \mathring{x}_3, \ldots\}$ of right-concatenation mappings, this set being indexed by the input set X_A. Then $\langle X_A^*\ \mathring{X}_A \rangle$ is a unary algebra, the "right-concatenation algebra over X_A^*", and the right invariance property can be expressed as the implication:

$\langle t_p\ t_q \rangle \in R_i,\ \langle t_p\ t'_p \rangle \in \mathring{x},\ \langle t_q\ t'_q \rangle \in \mathring{x} \Longrightarrow \langle t'_p\ t'_q \rangle \in R_i$

as shown in figure 2.11, where t_p and t_q are any tapes from $X_A{}^*$ and $\mathring{x}$ is an arbitrary right-concatenation mapping from $\mathring{X}_A$.

$$\begin{tikzcd}
t_p \arrow[r, "R_i"] \arrow[d, "\mathring{x}"] & t_q \arrow[d, "\mathring{x}"] \\
t'_p \arrow[r, phantom, "\Downarrow"] & t'_q \\
t'_p \arrow[r, "R_i"] & t'_q
\end{tikzcd}$$

Relation R_i is a congruence of $\langle X_A{}^*\ \mathring{X}_A \rangle$

Figure 2.11

Clearly equivalence R_i is preserved under each of the right-concatenation mappings $\mathring{x}$ from $\mathring{X}_A$, in which case R_i is a congruence of the right-concatenation algebra $\langle X_A{}^*\ \mathring{X}_A \rangle$. Then R_i can be used to define a quotient algebra of $\langle X_A{}^*\ \mathring{X}_A \rangle$, by considering the way the blocks of the preserved partition $X_A{}^*/R_i$ are assigned into each other under concatenation. More formally each input x from X_A has an associated mapping $\bar{x}^R$ over partition $X_A{}^*/R_i$ where

$\bar{x}^R = \left\{ \langle E\ E' \rangle \,\middle|\, E, E' \in X_A{}^*/R_i \quad \& \quad (E)\ \mathring{x} \subseteq E' \right\},$

and these mappings form an indexed set

$\bar{X}_R = \left\{ \bar{x}_1^R,\ \bar{x}_2^R, \ldots . \right\}$. The quotient algebra is then

$\langle X_A{}^*/R_i\ \overline{X}_R\rangle$, and the quotient algebra must be a homomorphic image of right-concatenation algebra $\langle X_A{}^*\ \overset{o}{X}_A\rangle$, where the homomorphism is the canonical mapping

$$H = \left\{\langle t\ E\rangle \,\middle|\, t \in X_A{}^*,\ E \in X_A{}^*/R_i \quad \& \quad t \in E\right\}$$

associated with partition $X_A{}^*/R_i$.

Any subset of $X_A{}^*$ will be called an "event", or more accurately a "$X_A{}^*$-event", and the preceeding shows that each state accessible from the reference state a_i defines an associated $X_A{}^*$-event, this being an equivalence class of partition $X_A{}^*/R_i$. The events associated with the states will be called "state events" with respect to the reference state a_i, and the unary algebra $\langle X_A{}^*/R_i\ \overline{X}_R\rangle$ will be called an "event semiautomaton". The event semiautomaton expresses the way state events interrelate under concatenation, and so the event semiautomaton is closely related to semiautomaton $A = \langle S_A\ \overline{X}_A\rangle$. To appreciate this consider figure 2.12, which illustrates an association $\langle a_j\ a'_j\rangle \in \bar{x}^A$.

(a) $\langle a_j\ a'_j\rangle \in \bar{x}^A$ (b) $\langle E_j\ E'_j\rangle \in \bar{x}^R$

Figure 2.12

Assuming state a_j to be accessible from reference state a_i then an equivalence class E_j of partition $X_A{}^*/R_i$ will correspond to state a_j, and then from previously $E_j \circ x \subseteq E'_j$ where E'_j is the equivalence class associated with a'_j. Therefore $\langle E_j\ E'_j \rangle \in \bar{x}^R$, and this can be represented as an arc labelled x from equivalence class E_j to equivalence class E'_j, as shown in figure 2.12(b). If the semiautomaton $A = \langle S_A\ \bar{X}_A \rangle$ is a_i-connected each state on the graph of semiautomaton A can be represented as an equivalence class of partition X_A^*/R_i, and each association $\langle a_j\ a'_j \rangle \in \bar{x}^A$ will be expressed as an association $\langle E_j\ E'_j \rangle \in \bar{x}^R$ between the corresponding equivalence classes. Then $A = \langle S_A\ \bar{X}_A \rangle$ and $\langle X_A^*/R_i\ \bar{X}_R \rangle$ will be isomorphic, and this is expressed in figure 2.13 as an isomorphism I relating event semiautomaton $\langle X_A^*/R_i\ \bar{X}_R \rangle$ to semiautomaton $\langle S_A\ \bar{X}_A \rangle$.

$$
\begin{tikzcd}
\langle X_A^*\ \overset{o}{X}_A \rangle \arrow[d, "H"'] \arrow[dr, "HI"] & \\
\langle X_A^*/R_i\ \bar{X}_R \rangle \arrow[r, "I"] & \langle S_A\ \bar{X}_A \rangle
\end{tikzcd}
$$

<u>Figure 2.13</u>

The figure also shows the homomorphism H relating concatenation algebra $\langle X_A^*\ \overset{o}{X}_A \rangle$ to the quotient algebra $\langle X_A^*/R_i\ \bar{X}_R \rangle$, and then HI must be a homomorphism of $\langle X_A^*\ \overset{o}{X}_A \rangle$ onto $\langle S_A\ \bar{X}_A \rangle$, and the semiautomaton $A = \langle S_A\ \bar{X}_A \rangle$ must be a homomorphic image of concatenation algebra $\langle X_A^* \overset{\circ}{X_A} \rangle$.

The preceeding shows that mapping $\Sigma a_i : X_A^* \longrightarrow S_A$ associates a state event with each state accessible from a_i, and that the state events are the equivalence classes associated with a congruence of the concatenation algebra $\langle X_A^* \overset{\circ}{X_A} \rangle$. The mapping $\Gamma a_i : X_A^* \longrightarrow Z_A$ associated with the reference state a_i, where by definition

$$\Gamma a_i = \left\{ \langle t \; z \rangle \,\middle|\, t \in X_A^*, \; z \in Z_A \quad \& \quad \langle a_i \; z \rangle \in \tilde{t}^A \right\},$$

can be analysed similarly and is illustrated in figure 2.14. For example the figure shows that output z_p is associated with a subset $M_p \subseteq X_A^*$, where each tape from M_p associates final-output z_p with the reference state a_i, in which case $M_p = [z_p] \, \Gamma a_i^{-1}$ or

$$M_p = \left\{ t \,\middle|\, t \in X_A^* \quad \& \quad \langle a_i \; z_p \rangle \in \tilde{t}^A \right\}$$

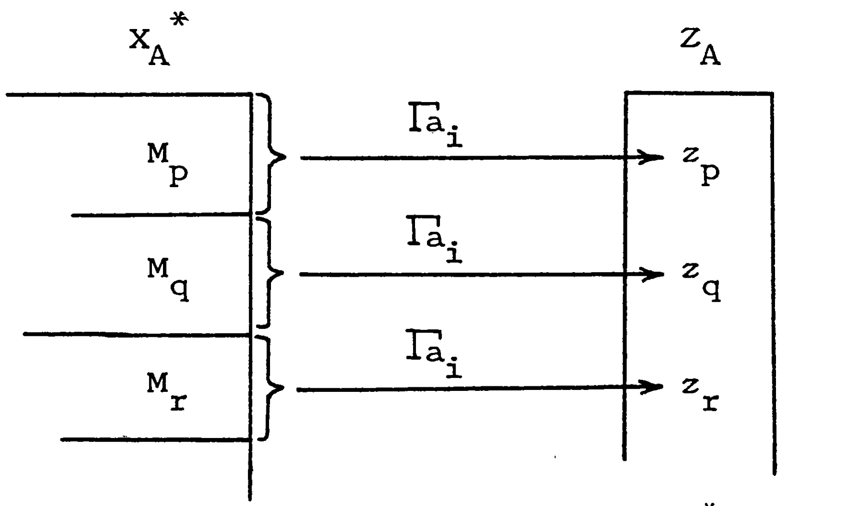


Figure 2.14 Mapping $\Gamma a_i : X_A^* \longrightarrow Z_A$

This subset M_p is the "output event" associated with z_p, and the figure shows that output z_q defines an output event $M_q = [z_q]\,\Gamma a_i^{-1}$, that output z_r defines an output event M_r, and so on. In general some output symbols will not define a nonvoid output event, however, since the final-outputs associated with state a_i by tapes from X_A^* might not exhaust Z_A. Alternatively the kernel $\ker(\Gamma a_i)$ of mapping Γa_i is an equivalence over the domain $D[\Gamma a_i]$, where $D[\Gamma a_i] = X_A^*$ since automaton $\hat{A}$ is complete, and $\ker(\Gamma a_i)$ associates an equivalence class with each element of the codomain $C[\Gamma a_i] \subseteq Z_A$.

Considering in particular the output event M_p defined by output symbol z_p, any tape t from M_p associates output symbol z_p with state a_i, that is $\langle a_i\ z_p\rangle \in \tilde{t}^A$. However $\tilde{t}^A = \bar{t}^A \omega_A$ for a Moore automaton, and so $\langle a_i\ z_p\rangle \in \tilde{t}^A$ implies that $\bar{t}^A$ assigns a_i to some final-successor a_i' where $\langle a_i'\ z_p\rangle \in \omega_A$. The mapping ω_A might assign several states $a_u, a_v, a_w, \ldots\ldots \in S_A$ to the output z_p, in which case either $\langle a_i\ a_u\rangle \in \bar{t}^A$, or $\langle a_i\ a_v\rangle \in \bar{t}^A$, or $\langle a_i\ a_w\rangle \in \bar{t}^A$, and so on, so if $E_u, E_v, E_w, \ldots\ldots$ are the state events associated with these states then $t \in E_u \cup E_v \cup E_w \cup \ldots\ldots$, therefore $M_p \subseteq E_u \cup E_v \cup E_w \cup \ldots\ldots$ Assuming conversely $t \in E_u \cup E_v \cup E_w \cup \ldots\ldots$ then $\langle a_i\ a_i'\rangle \in \bar{t}^A$ for some a_i' where $\langle a_i'\ z_p\rangle \in \omega_A$, because each of the states $a_u, a_v, a_w, \ldots\ldots$ is assigned by ω_A to z_p, therefore $\langle a_i\ z_p\rangle \in \bar{t}^A\,\omega_A$, that is $\langle a_i\ z_p\rangle \in \tilde{t}^A$, so $t \in M_p$. Consequently $M_p = E_u \cup E_v \cup E_w \cup \ldots\ldots$, showing that output event M_p is a union of certain state events, these being the state events defined by the states associated with z_p by output mapping ω_A.

Clearly a similar argument can be applied to each of the output events associated with mapping $\overline{I}a_i$, so that each output event is a union of certain state events, and is therefore a union of certain equivalence classes defined by a congruence of $\langle X_A^* \overset{o}{X_A} \rangle$. In fact this can be confirmed by taking a_i to be the "initial state" of the automaton, defining an appropriate set of "distinguished" states and considering the tapes assigning a_i to some distinguished state. Such tapes are said to be "recognised" or "accepted" by the automaton, and adopting this approach has simplified a result originated by Nerode.

Result - The Nerode Theorem [Nerode; Rabin & Scott]

For any X^*-event M, the following three conditions are equivalent:

(i) a finite automaton accepting M can be defined,

(ii) M is the union of some of the equivalence classes of a right-invariant equivalence relation over X^* with a finite number of equivalence classes,

(iii) the right-invariant equivalence relation E over X^*, where

$$E = \left\{ \langle x\ y \rangle \middle| x, y \in X^* \quad \& \quad (\forall z)(x \circ z \in M \Longleftrightarrow y \circ z \in M) \right\},$$

has a finite number of equivalence classes.

In particular let $\{a_u, a_v, a_w, \ldots \}$ be the set of the distinguished states, these being the states assigned to z_p by ω_A, and take a_i as the initial state. Then any tape t assigning a_i to one of the distinguished states will associate z_p with a_i, for example if $\langle a_i \; a_u \rangle \in \bar{t}^A$ then $\langle a_i \; z_p \rangle \in \tilde{t}^A$, since $\tilde{t}^A = \bar{t}^A \omega_A$ and $\langle a_u \; z_p \rangle \in \omega_A$. Conversely, if $t \in M_p$ then t must assign a_i to one of the distinguished states. Clearly this choice of initial and distinguished states has defined an automaton accepting event M_p, and by the Nerode theorem $M_p = E_u \cup E_v \cup E_w \cup \ldots.$ since each of the states $a_u, a_v, a_w, \ldots.$ is assigned to z_p by ω_A.

The Nerode theorem is particularly important in circuit synthesis, since a finite automaton accepting an event M can be formalised so long as M can be expressed as a union of equivalence classes of a finite right-invariant equivalence over X^*. Such a right-invariant equivalence can be formalised as an event semiautomaton, where the equivalence classes become the state events, and this can be used as a basis for designing a sequential circuit. It will often be difficult, however, to determine an appropriate right-invariant equivalence, and to approach this problem it is necessary to consider a further property of the state events associated with finite automata. In fact the state events, and consequently the output events, are "regular", that is they can be represented as "regular expressions".

<u>Definition</u>

<u>Regular expressions</u> over a set X are defined recursively:

(i) the elements of X are regular expressions over X, as are λ and the void symbol ϕ,

(ii) if F and G are regular expressions over X then so are $F \circ G$, $f(F,G)$ and F^*, where $f(F,G)$ denotes any Boolean function of F and G and $F^* = \lambda \cup F \cup F^2 \cup F^3 \cup \ldots$,

(iii) nothing else is a regular expression over X unless its being so follows from a finite number of applications of (i) and (ii) above.

An event defined as a regular expression is a <u>regular event</u>.

For example it has been shown that the kernel $\ker(\Sigma a_i)$ of mapping $\Sigma a_i : X_A^* \longrightarrow S_A$ is an equivalence over X_A^*, where the tapes forming an equivalence class associate a common successor with the reference state a_i. In fact each of these equivalence classes, that is each state-event of the event semiautomaton $\langle X_A^*/R_i\ \overline{X}_R \rangle$, is a regular event.

<u>Result</u> [Kohavi]

The set of tapes associating a common successor with an arbitrary state of a finite automaton is a regular event.

2.7 Conclusion

The prime aim has been the development of useful definitions and symbology, furthermore various unifying concepts have been presented. The homomorphism and congruence concepts have been of particular value, firstly in relation to the useful identity $\overline{t_p \circ t_q}^A = \overline{t}_p^A \cdot \overline{t}_q^A$, which was used to show that the indexing from T_A to $\overline{T}_A$ is a homomorphism of the semigroup $\langle T_A \circ \rangle$ onto the semigroup $\langle \overline{T}_A \cdot \rangle$. Secondly, the equivalence R_i was shown to be preserved under the right-concatenation mappings $\overset{\circ}{x}$, and R_i was interpreted as a congruence of concatenation algebra $\langle X_A^* \overset{\circ}{X}_A \rangle$.

However "state events", "event semiautomata" and the Nerode theorem are of particular importance. It has been shown that a set of input tapes or a "state event" is associated with each accessible state, and that the tapes forming a state event associate a common final-successor with the automaton reference state. This is especially important in understanding the meaning of the state-codes associated with sequential circuits. An observer of a sequential circuit, having read off the existing state-code, can make a deduction regarding the circuit since the state-code represents a set of input sequences, or represents a "regular event". In a sense the state-code of a sequential circuit represents the various ways the circuit could have been left in that state, or represents an "ambiguity" regarding the input sequence applied in the past. Assuming the circuit is of Moore type, the output logic will transform the state-codes to output codes and can be regarded as an association of output codes with regular events.

Furthermore the way the state events interrelate under concatenation can be expressed as an "event semiautomaton", where such a semiautomaton has "state events" instead of states, and represents a right-invariant equivalence. This is a particularly important concept in sequential circuit synthesis, and to appreciate this consider the design of a circuit so that a particular output code is associated with a particular set of input sequences. Then these input sequences can be considered to form an "objective event", and it follows from the Nerode theorem that the design can proceed by formalising an event semiautomaton so that the objective event can be expressed as a union of certain state events. The design can then be completed by representing the state events as state-codes, and designing combinational circuitry to associate the desired output code with the state events forming the union.

There are, however, various complications to this approach, firstly because an appropriate right-invariant equivalence must be found, and secondly since multiple outputs must sometimes be accommodated. Furthermore the design must take into account the environment with which the proposed circuit is to interface, for example the system providing inputs for the proposed circuit might be incapable of delivering certain input sequences, and the response of the proposed circuit to these sequences is then immaterial. It is intended to cover these considerations in the next chapter, and to show that event semiautomata, the Nerode theorem and regular expressions combine to give a rigourous approach to sequential circuit design.

CHAPTER THREE: Objective Specification

3.1 Introduction

The input and output conditions associated with a proposed discrete-parameter system can be expressed as appropriate finite sets of arbitrary symbols, an "input set" such as $X = \{x_1, x_2, x_3, \ldots\}$ being defined in representation of the system input conditions, and an "output set" $Z = \{z_1, z_2, z_3, \ldots\}$ being defined to represent the system output conditions. The represented conditions might be conceptual, such as "true" and "false", however it may be that the system is intended to interface with an existing coded-information environment, in which case the represented conditions will be codes.

The desired performance of the proposed system is often expressed informally, in the form of rules associating a particular output condition with particular sequences of input conditions. Once these rules are expressed using the symbology of the sets X and Z, the desired performance can be formalised as an "objective mapping" $\Gamma obj : X^* \longrightarrow Z$. To develop this by example, consider the following objective statement with input set $X = \{x_1, x_2\}$ and output set $Z = \{z_1, z_2, z_3\}$.

Objective Statement

(a) Properties of the input environment

Occurrences of input symbol x_1 must never be adjacent.

(b) Desired performance

Output symbol z_1 is to be associated with any valid tape ending with the sequence $\langle x_1\ x_2 \rangle$, and output symbol z_2 is to be associated with any valid tape ending with the sequence $\langle x_2\ x_2 \rangle$.

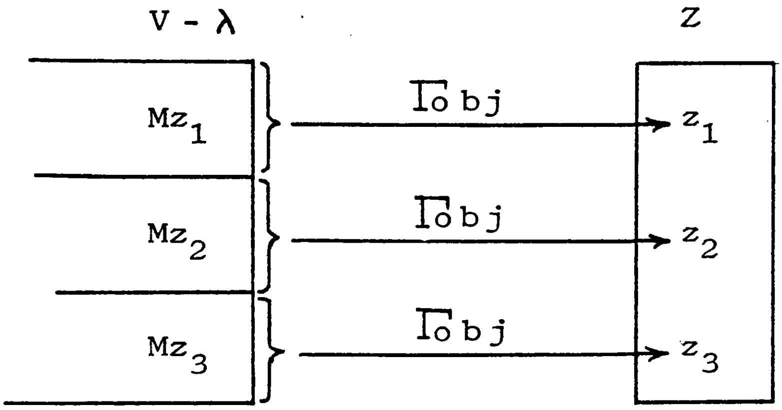
(c) Properties of the output environment

Any output symbol can be associated with any tape of unit length.

Such an objective statement has three aspects, the first declaring as "valid" only those tapes representing input sequences to be presented at the system input. In the example, the environment supplying inputs to the proposed system will never present consecutive occurrences of the condition represented as x_1, so a "valid" tape cannot have adjacent occurrences of x_1, for example $\langle x_1\ x_2\ x_1 \rangle$ is a valid tape whereas $\langle x_1\ x_1\ x_2 \rangle$ is invalid. The set of the valid tapes, that is the subset of X^* where each tape is consistent with the input environment, will be called the "valid event" and will be denoted V. The null tape λ is included as a valid tape, that is $\lambda \in V$, and $\langle x_1\ x_2\ x_1 \rangle \in V$, for example, whereas $\langle x_1\ x_1\ x_2 \rangle \notin V$.

The second aspect of the objective statement associates an output symbol with certain valid tapes, and this assignment can be used to formalise an "objective mapping" $\Gamma obj : X^* \longrightarrow Z$ with domain $D[\Gamma obj] = V - \lambda$ and codomain $C[\Gamma obj] = Z$. For convenience any tape t such that $t = t_p \circ t_q$, for some tapes t_p and t_q (perhaps $t_p = \lambda$), will be said to have "final subtape" t_q, and the objective mapping Γ_{obj} is defined so that any tape from $V - \lambda$ with final subtape $\langle x_1\ x_2 \rangle$ is assigned by Γ_{obj} to z_1, and tapes from $V - \lambda$ with final subtape $\langle x_2\ x_2 \rangle$ are assigned by Γ_{obj} to z_2. The output to be associated with any other valid tape is unspecified, however care must be taken to ensure that such a tape is not assigned to z_1 or to z_2. Each of these output symbols is being used to signify a particular event, for example z_1 is to be associated with the event formed by the valid tapes ending $\langle x_1\ x_2 \rangle$, so z_1 must not be associated with any other type of tape. This has been anticipated in defining output set $Z = \{z_1, z_2, z_3\}$, and whenever a tape from $V - \lambda$ cannot be assigned to z_1 or z_2 this tape is assigned by Γ_{obj} to the "neutral" output symbol z_3.

Disregarding for the present part (c) of the objective statement, the objective mapping $\Gamma_{obj} : X^* \longrightarrow Z$ can be visualised as in figure 3.1.



<u>Figure 3.1</u> <u>Objective mapping Γ_{obj}</u>

The kernel ker $(\Gamma obj) = \Gamma obj . \Gamma obj^{-1}$ of the objective mapping is an equivalence over the domain $V - \lambda$, with an equivalence class for each element of the codomain Z, and the figure shows that the output symbols z_1, z_2 and z_3 are associated with respective equivalence classes Mz_1, Mz_2 and Mz_3, so Mz_1 is the set of the tapes from $V - \lambda$ with final subtape $\langle x_1\ x_2 \rangle$, Mz_2 is the set of the tapes from $V - \lambda$ with final subtape $\langle x_2\ x_2 \rangle$, and the remaining tapes from $V - \lambda$ form Mz_3. These sets Mz_1, Mz_2 and Mz_3, that is the equivalence classes defined by the objective mapping, will be called the "objective output events", furthermore each of these events can be defined as a regular expression. For example any valid tape with final subtape $\langle x_1\ x_2 \rangle$, and with just one occurrence of x_1, must be of the form $x_2^* \ x_1 \ x_2$, so that $\langle x_1\ x_2 \rangle$ is preceeded by a succession of occurrences of x_2, or $\langle x_1\ x_2 \rangle$ is without any preceeding symbols. More accurately, this set of tapes is defined by the regular expression $x_2^* \circ x_1 \circ x_2$, however in using regular expressions it is convenient to omit the concatenation operator. Similarly any valid tape with final subtape $\langle x_1\ x_2 \rangle$ and with just two occurrences of x_1 must be of the form $x_2^* \ x_1 \ x_2 \ x_2^* \ x_1 \ x_2$, that is $\langle x_1\ x_2 \rangle$ preceeded by at least one occurrence of symbol x_2 (since occurrences of x_1 cannot be adjacent), preceeded by an occurrence of x_1 following any number of occurrences of x_2. Then the regular expression $x_2^* \ x_1 \ x_2 \ x_2^* \ x_1 \ x_2 \ x_2^* \ x_1 \ x_2$ defines the set of all the valid tapes with final subtape $\langle x_1\ x_2 \rangle$ and with just three occurrences of x_1, and it is evident that the regular expression $(x_2^* x_1 x_2)^* x_2^* x_1 x_2$ defines the set of all the valid tapes with final subtape $\langle x_1 x_2 \rangle$, since

$(x_2^* x_1 x_2)^* x_2^* x_1 x_2 =$

$$x_2^* x_1 x_2 \cup x_2^* x_1 x_2 x_2^* x_1 x_2 \cup \ldots\ldots$$

Hence $Mz_1 = (x_2^* x_1 x_2)^* x_2^* x_1 x_2$, it being evident from this identity that no distinction will be made between a regular event and the corresponding regular expression. By similar reasoning,

$$Mz_2 = x_2 x_2^* x_2 \cup (x_2^* x_1 x_2)^* x_2^* x_1 x_2 x_2^* x_2 \quad \text{and}$$

$$Mz_3 = x_2 \cup (x_2^* x_1 x_2)^* x_2^* x_1.$$

This shows that each of the objective output events is regular, in which case a finite automaton accepting Mz_1, for example, can be defined [Kleene], and similarly for the regular events Mz_2 and Mz_3. The present aim, however, is to form a single automaton with three separate sets of distinguished states, a set of distinguished states being associated with each of the output symbols z_1, z_2 and z_3. When the distinguished states associated with z_1 are considered the automaton must accept the regular event Mz_1, and similarly for the events Mz_2 and Mz_3, so with an appropriate choice of distinguished states the automaton will accept each of the objective output events. The automaton will then provide a "finite-state" expression of the objective mapping, since Γ_{obj} will be one of the state-mappings, and this will be useful in designing a sequential circuit with multiple outputs.

3.2 Finite-state expression

It has been seen that finite automata and regular expressions are closely related, in fact various formal procedures can be used to progress from a given regular expression to a finite automaton accepting the corresponding regular event [Ott & Feinstein; McNaughton & Yamada; Brzozowski]. In principle each of these procedures accommodates multiple outputs, however the use of "improper state diagrams" [Ott & Feinstein] for the present design example will prove particularly illustrative. The integrity of this approach is established since any regular expression can be represented as an improper state diagram [Ott & Feinstein; Kohavi], and an improper state diagram defines an automaton recognising the corresponding regular event [Rabin & Scott].

The interpretation of the improper state diagrams is usually vague, for example the graphs are usually viewed as "nondeterministic automata ", and arcs labelled λ are interpreted as "instantaneous" state transitions. A sound interpretation of the graphs can be given with reference to the Nerode theorem. Each graph node represents a regular event, and the graph arcs express the way the events are related under concatenation, for example an arc labelled x from a regular event E_i on the graph to a regular event E_j expresses $E_i \circ x \subseteq E_j$, so the graphs are closely related to event semiautomata. In particular an arc labelled λ from E_i to E_j expresses $E_i \circ \lambda \subseteq E_j$, that is $E_i \subseteq E_j$, and the aim is to form a graph so that each objective output event can be expressed as a union $E_i \circ \lambda \cup E_j \circ \lambda \cup \ldots$, where $E_i, E_j, \ldots$ are represented as graph nodes. To retain close relationship with regular expressions it is initially conceded, for an event such as E_i, that $E_i \circ x$ can be a subset of several events on the graph. Then several arcs from E_i will be labelled x, and the graph will be "nondeterministic" or "improper".

Any event represented on a graph will be said to be "generated" by the graph, and a given regular expression can be used to form a graph generating the associated regular event. In the case of a regular expression using the union, concatenation and iteration operators only, the graph can be formed by combining simple graphs. For example figure 3.2(a) shows a node representing the regular event λ and a node representing the regular event x_1, the corresponding regular expressions being shown against each node. In fact the graph illustrates $\lambda \circ x_1 = x_1$, or shows that the concatenation of event $\{\lambda\}$ by input symbol x_1 forms the event $\{x_1\}$, and in a similar way the "regular event graph" of figure 3.2(b) generates the regular event x_2. If now event λ of graph (b) is replaced by event x_1 of graph (a), the resulting regular event graph of figure 3.2(c) generates the event $x_1 \circ x_2$, in fact the graph illustrates $(\lambda \circ x_1) \circ x_2 = x_1 \circ x_2$. This "concatenation construction" extends in a natural way to

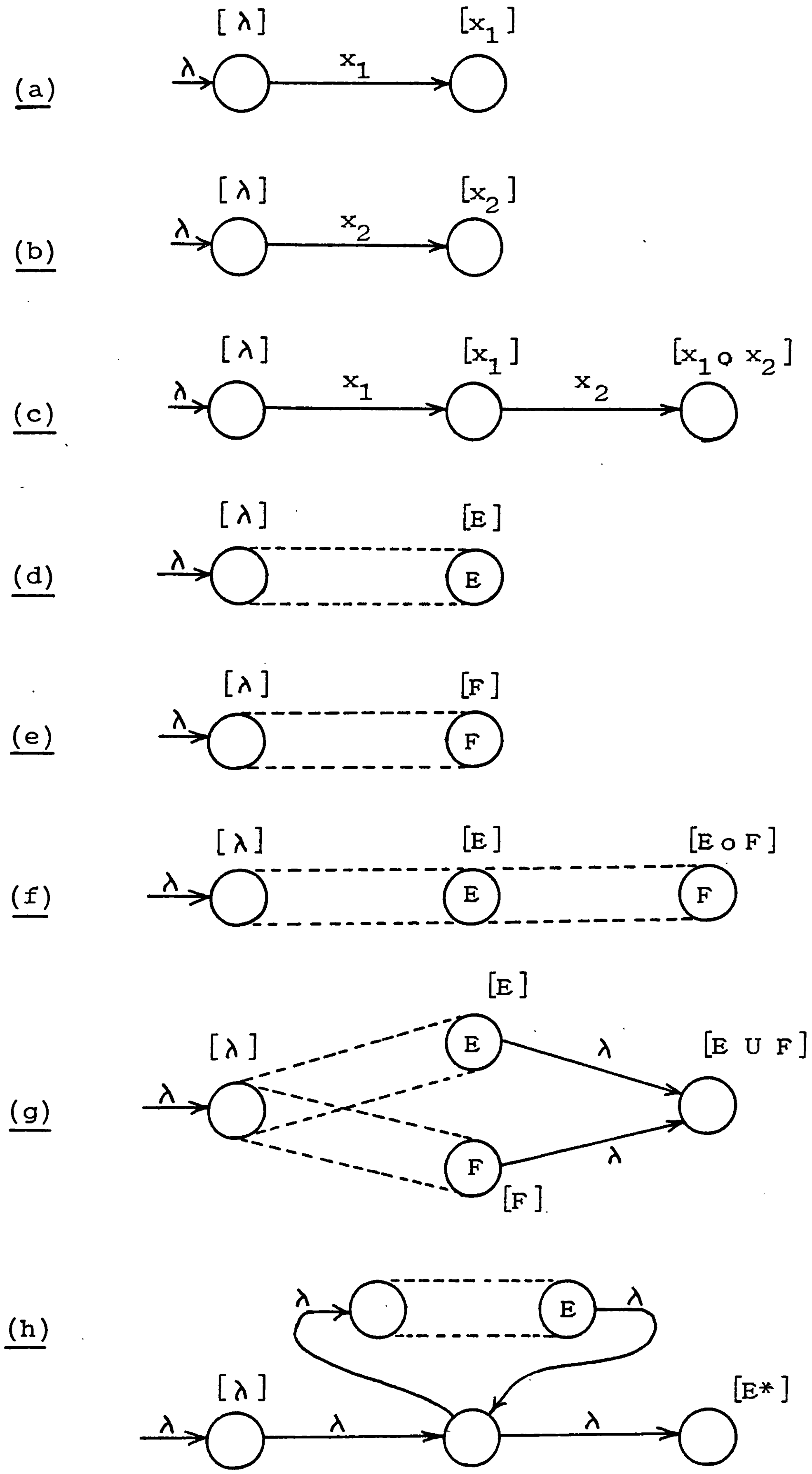


Figure 3.2 Constructions using regular-event graphs

larger graphs, so that a graph generating a regular event E, as represented informally in figure 3.2(d), can be combined with the similar graph of figure 3.2(e) to produce a graph generating the regular event $E \circ F$, as shown in figure 3.2(f). Alternatively, these graphs can be combined by "union construction" to produce graph (g) generating the regular event E U F, the construction being based on the identity $E \circ \lambda \cup F \circ \lambda = E \cup F$. Finally the "iteration construction" is illustrated by figure 3.2(h), the regular event E^* being generated by forming a loop around the graph generating the regular event E. Clearly the constructions introduce surplus λ arcs, however these are easily removed once the construction process is complete.

These formal constructions can be used to produce a graph for any regular expression involving union, concatenation and iteration operators, however the constructions can become complex, even for modest regular expressions. For example the graph generating the objective output event $Mz_1 = (x_2^* \; x_1 \; x_2)^* \; x_2^* \; x_1 \; x_2$ would involve 15 nodes and 11 surplus λ arcs. In practice a less formal approach can be adopted, and this will be demonstrated in deriving a graph to generate the above event Mz_1. The formal constructions would produce the graph of figure 3.3(a) to generate the regular event x_2^*, and would then produce the graph of figure 3.3(b) to generate the event $x_2^* \; x_1 \; x_2$, however the simpler graph (c) is based on the observation $\lambda \cup x_2^* = x_2^*$, and generates the same event. In fact the graph illustrates

$((\lambda \cup x_2^*) \circ x_1) \circ x_2 = x_2^* \circ x_1 \circ x_2$, and it is also useful to observe that any tape of the form $x_2^* \ x_1 \ x_2$, consider for example the tape $\langle x_2 \ x_2 \ x_1 \ x_2 \rangle$, defines a succession of arcs from the node associated with λ to the node representing the regular event $x_2^* \ x_1 \ x_2$. Since graph (c) generates the regular event $x_2^* \ x_1 \ x_2$, iteration construction can be used to produce a graph generating the regular event $(x_2^* \ x_1 \ x_2)^*$ as in figure 3.3(d). However one of the λ arcs can be recognised as redundant, and the simpler graph of figure 3.3(e) is obtained. Now graph (e) generates the event $(x_2^* \ x_1 \ x_2)^*$, and graph (c) generates the event $x_2^* \ x_1 \ x_2$ so concatenation construction produces the graph of figure 3.3(f), which generates the objective output event $Mz_1 = (x_2^* \ x_1 \ x_2)^* \ x_2^* \ x_1 \ x_2$. The output event generated by the graph is distinguished by double circles.

Having produced a graph generating the objective output event Mz_1, the same approach can be used to produce a graph generating Mz_2, and can then be used to produce a graph generating Mz_3. The three graphs can be visualised as in figure 3.4(a), and then the composite graph of figure 3.4(b), formed by a variant of the union construction, can be derived and generates all three objective output events. This approach is close to that suggested by Ott and Feinstein, however a more efficient approach is possible since care has been taken to define the objective output events using common phrases.

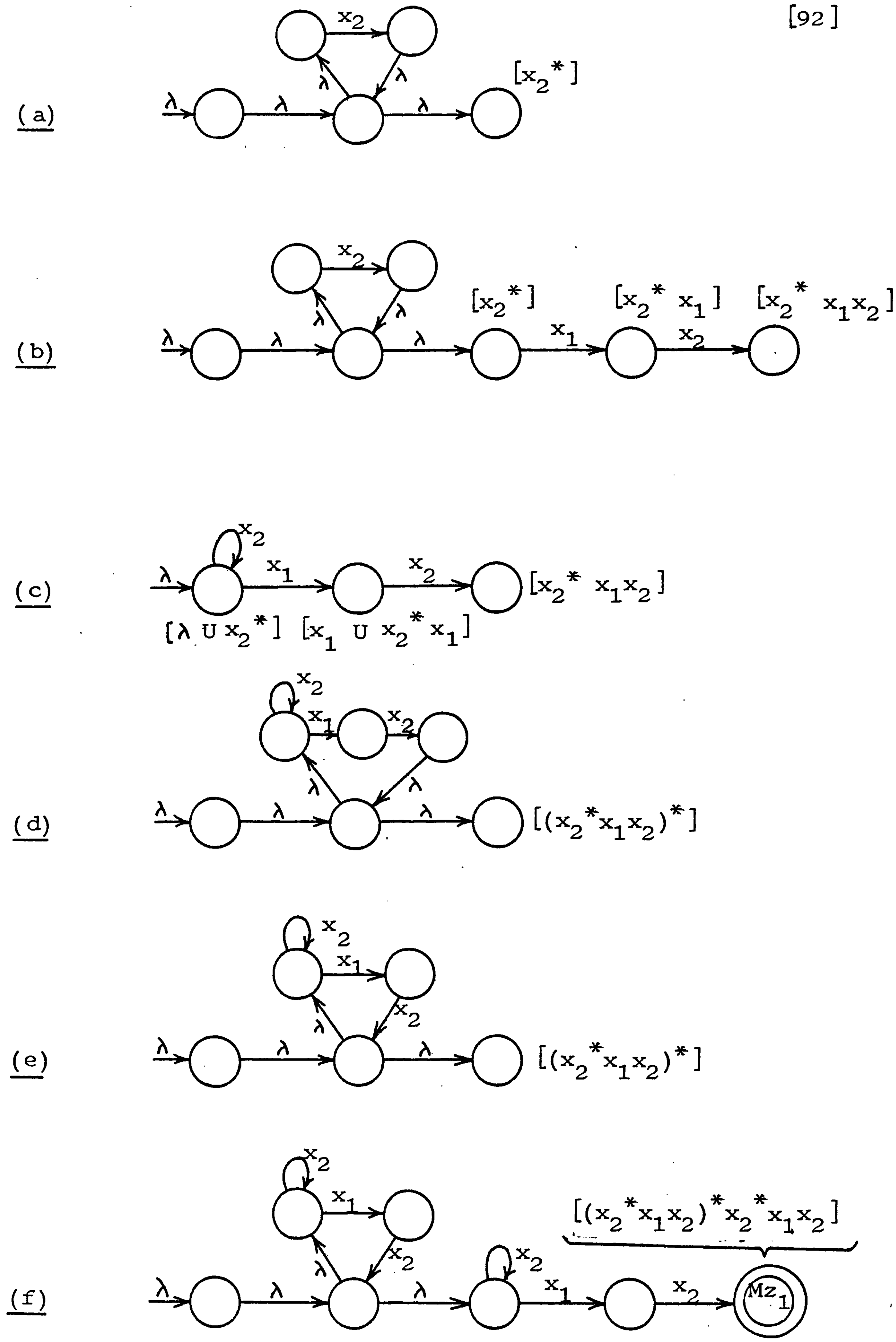
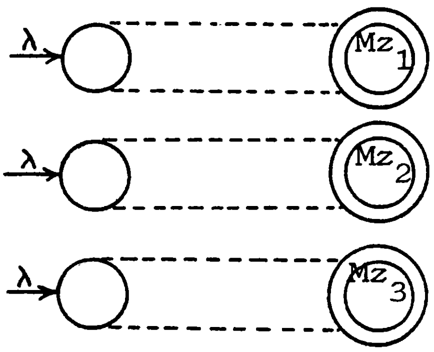
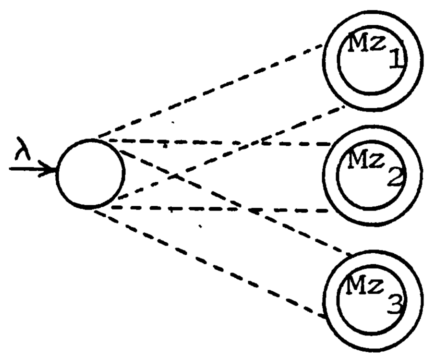

(a)
[x2*]
(b)
[x2*]
[x2* x1]
[x2* x1x2]
(c)
[x2* x1x2]
[λ U x2*]
[x1 U x2* x1]
(d)
[(x2*x1x2)*]
(e)
[(x2*x1x2)*]
(f)
[(x2*x1x2)*x2*x1x2]
Mz1

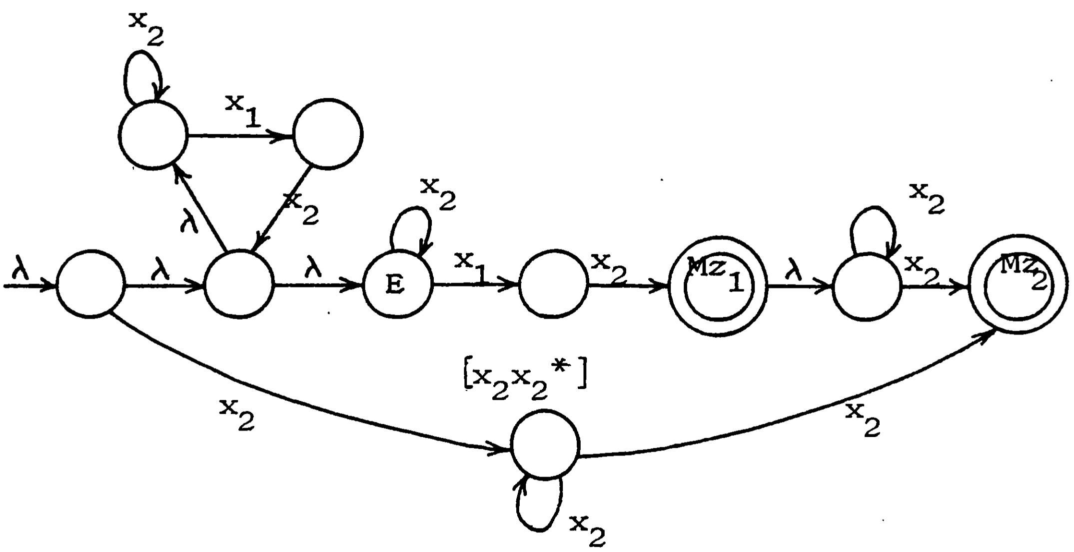
Figure 3.3



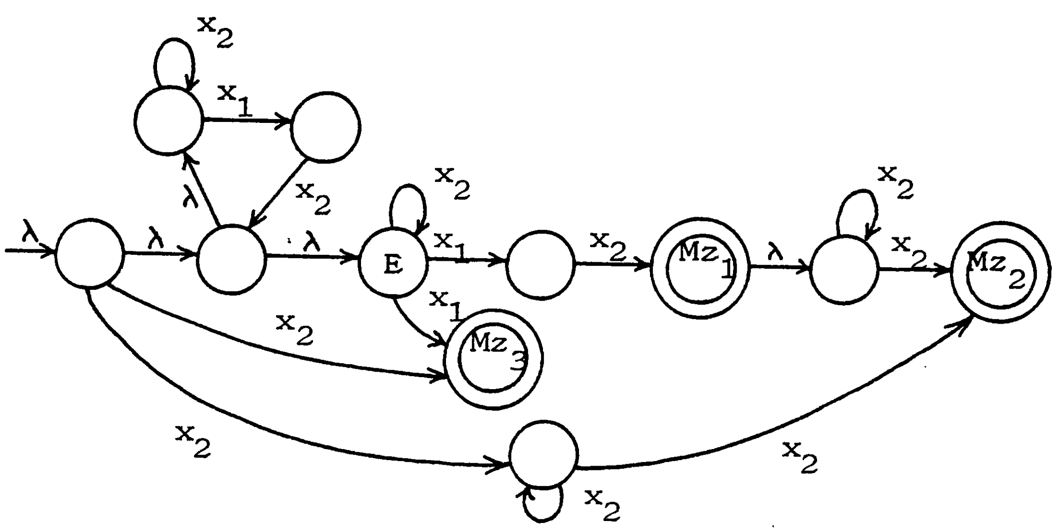
(a)



(b)



(c)



(d) Regular-event graph generating Mz_1, Mz_2 and Mz_3

Figure 3.4

Specifically $Mz_1 = (x_2^* x_1 x_2)^* x_2^* x_1 x_2$

and $Mz_2 = x_2 x_2^* x_2 \cup (x_2^* x_1 x_2)^* x_2^* x_1 x_2 x_2^* x_2$

so $Mz_2 = x_2 x_2^* x_2 \cup Mz_1 \circ x_2^* x_2$

and this is expressed in the graph of figure 3.4(c), which then generates Mz_1 and Mz_2. Furthermore the node E on the graph represents the regular event $(x_2^* x_1 x_2)^* x_2^*$, and $Mz_3 = x_2 \cup (x_2^* x_1 x_2)^* x_2^* x_1$

so $Mz_3 = x_2 \cup E \circ x_1$

as shown in figure 3.4(d). This graph generates each of the objective output events Mz_1, Mz_2 and Mz_3, and is "improper" in the sense that the arcs from each node are not always distinctly labelled. For example two arcs from the event E are labelled x_1, since $E \circ x_1$ is a subset of two events represented on the graph.

The surplus λ arcs on the graph can now be removed, an arc labelled λ from an event E_i to an event E_j being replaced by arcs from E_i in duplication of those from E_j. The graph of figure 3.4(d) has four surplus λ-arcs, and these are distinguished in figure 3.5(a) although of course $\lambda(1) = \lambda(2) = \lambda(3) = \lambda(4) = \lambda$. Disregarding, for the present, the removal of $\lambda(4)$, the graph of figure 3.5(b) is formed from figure 3.5(a) by omitting the $\lambda(1)$, $\lambda(2)$ and $\lambda(3)$ arcs. Now the $\lambda(1)$ arc is replaced by arcs from event Mz_1 in duplication of those from the event G, that is $\lambda(1)$ is replaced by an arc labelled x_2 from Mz_1 to G and an arc labelled x_2 from Mz_1 to Mz_2, as shown in figure 3.5(c).

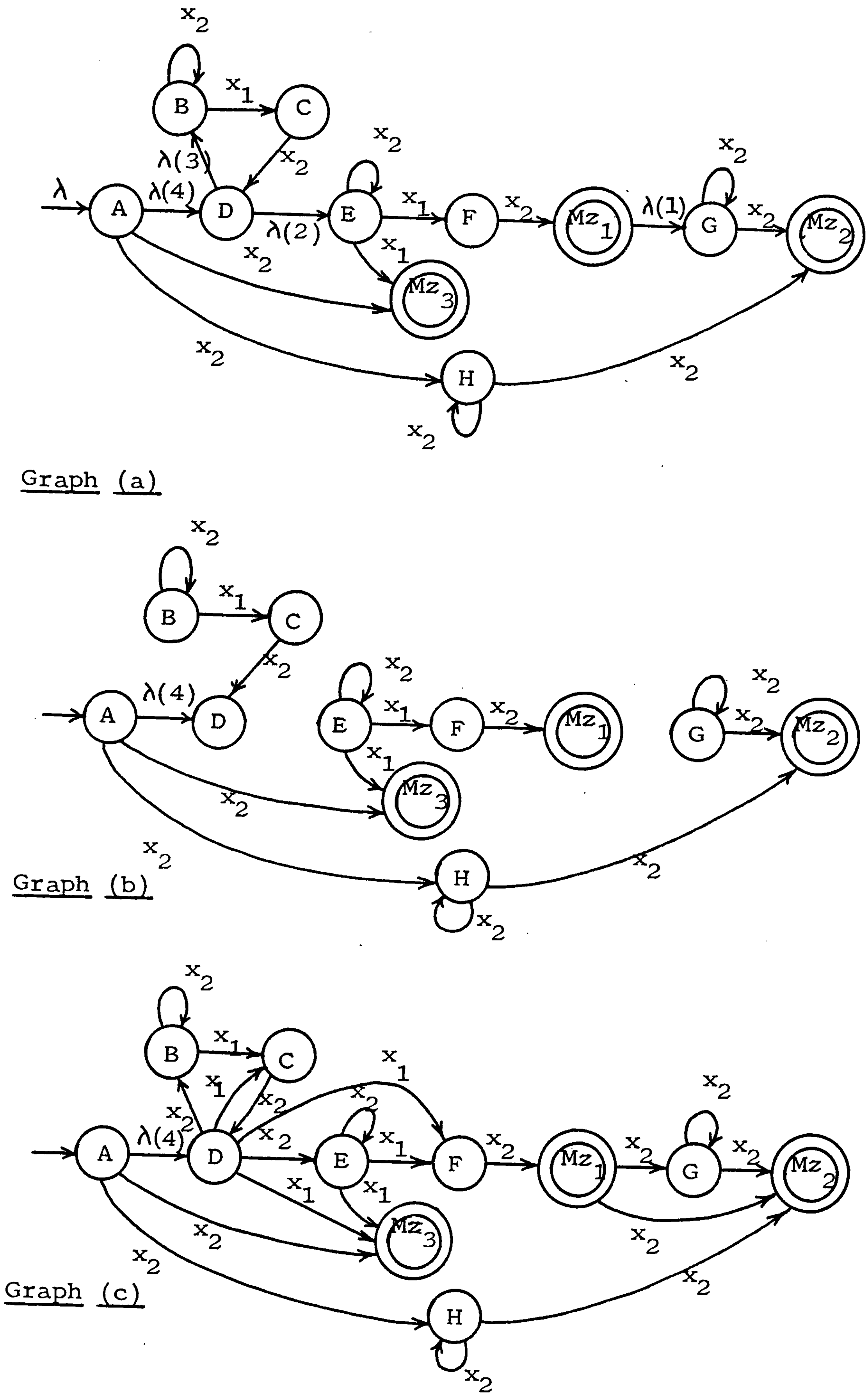
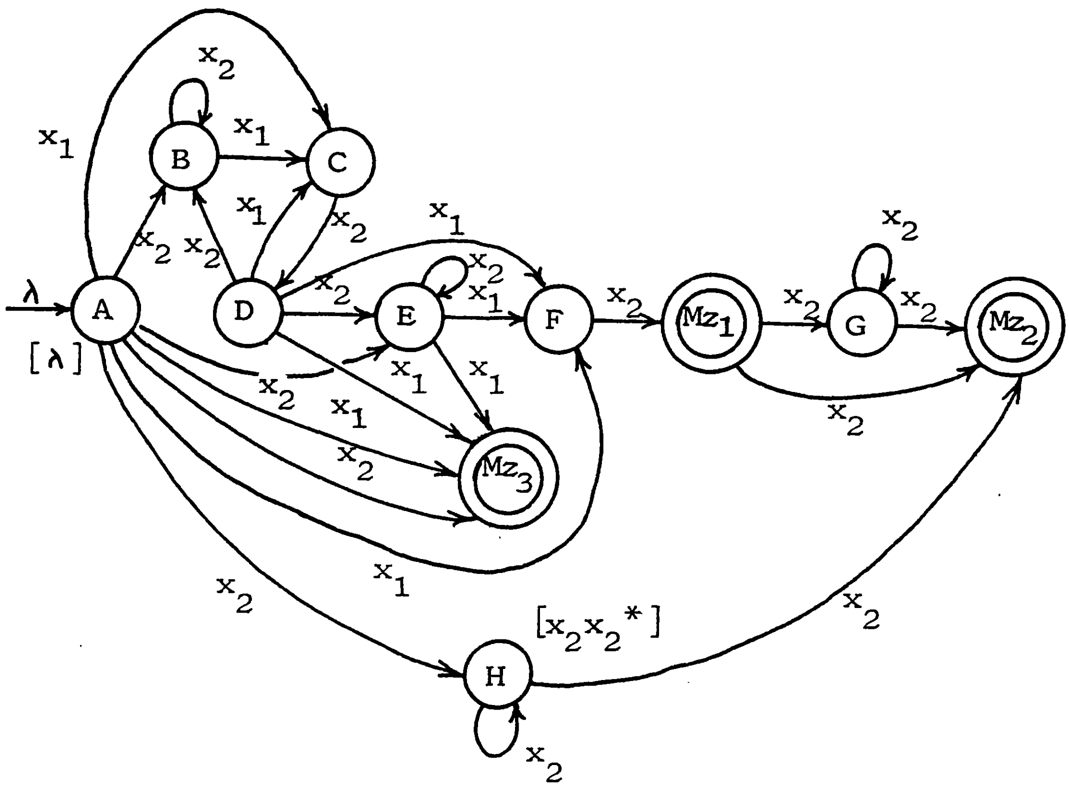


Figure 3.5

This graph is completed by introducing arcs to replace $\lambda(2)$ and $\lambda(3)$, and once $\lambda(3)$ is removed the arc $\lambda(4)$ can be considered. This arc could not be removed from graph (a) directly, since $\lambda(4)$ terminates on the event D and $\lambda(2)$ originates from this event, as also does $\lambda(3)$. The arc $\lambda(4)$ can be removed from graph 3.5(c), however, to produce the graph of figure 3.6.



<u>Figure 3.6</u>

As before each node represents a regular event, for example the regular event A is defined by the regular expression λ , and event H is defined as $x_2\, x_2^*$. The arcs express the way the events interrelate under concatenation, for example the x_1 arc from B to C expresses $B \circ x_1 \subseteq C$, and the graph shows that each of the objective output events can be defined as a union of certain regular events. For example the graph shows Mz_3 to be the union of the regular events $A \circ x_1$, $A \circ x_2$, $D \circ x_1$ and $E \circ x_1$. Clearly the events represented on the graph do not form a partition, for example $D \circ x_2$ is a subset of B and is also a subset of E. In fact the events form a cover $\psi = \left\{ A, B, \ldots .\ H, Mz_1, Mz_2, Mz_3 \right\}$ of the valid event V, and the cover is "right invariant" over V. For any block P of cover ψ , and for any input symbol $x \in X$, either $P \circ x$ consists entirely of invalid tapes or $P \circ x \subseteq P'$ for some cover block P'. Alternatively, consider the compatibility relation R_ψ associated with cover ψ , where $t_p R_\psi t_q$ expresses that the tapes t_p and t_q belong to a common block of cover ψ . Assuming $t_p R_\psi t_q$, and assuming $x \in X$, then either $t_p \circ x, t_q \circ x \notin V$ or $t_p \circ x \;\; R_\psi \; t_q \circ x$. Then

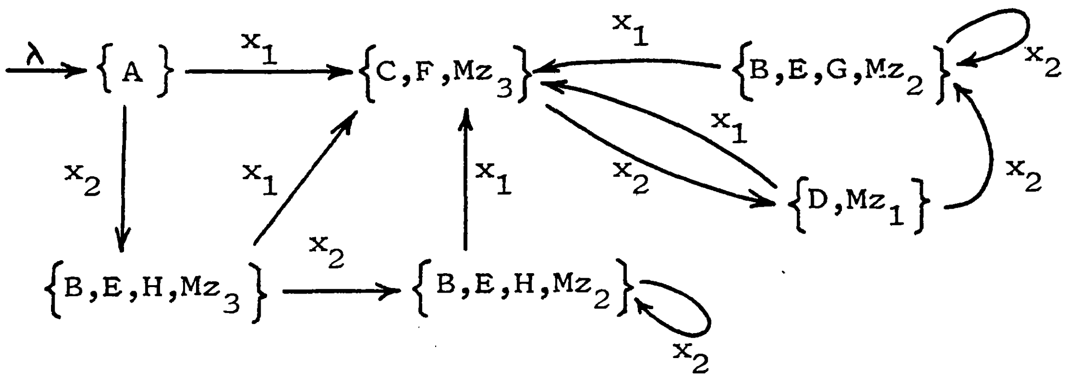
$t_p R_\psi t_q,\ t_p \circ x \in V,\ t_q \circ x \in V$ implies $t_p \circ x \, R_\psi \, t_q \circ x$,

so relation R_ψ is "preserved over V".

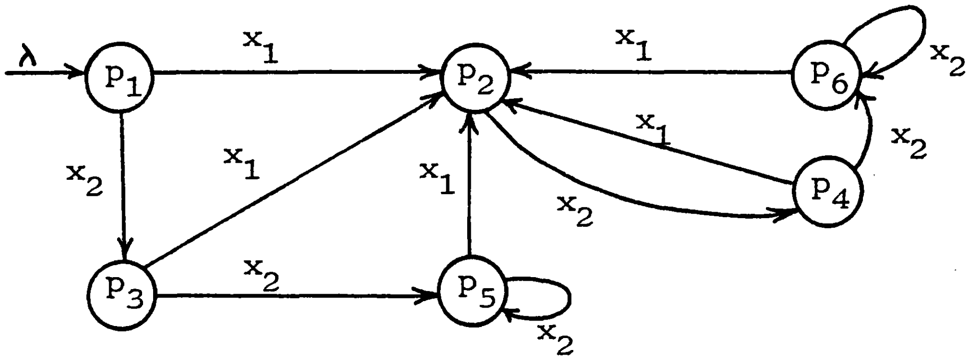
The real significance of the graph, however, is the relationship with the Nerode theorem. A finite right-invariant equivalence over valid event V can be derived

	x_1	x_2
$\{A\}$	$\{C,F,Mz_3\}$	$\{B,E,H,Mz_3\}$
$\{C,F,Mz_3\}$	-	$\{D,Mz_1\}$
$\{B,E,H,Mz_3\}$	$\{C,F,Mz_3\}$	$\{B,E,H,Mz_2\}$
$\{D,Mz_1\}$	$\{C,F,Mz_3\}$	$\{B,E,G,Mz_2\}$
$\{B,E,H,Mz_2\}$	$\{C,F,Mz_3\}$	$\{B,E,H,Mz_2\}$
$\{B,E,G,Mz_2\}$	$\{C,F,Mz_3\}$	$\{B,E,G,Mz_2\}$

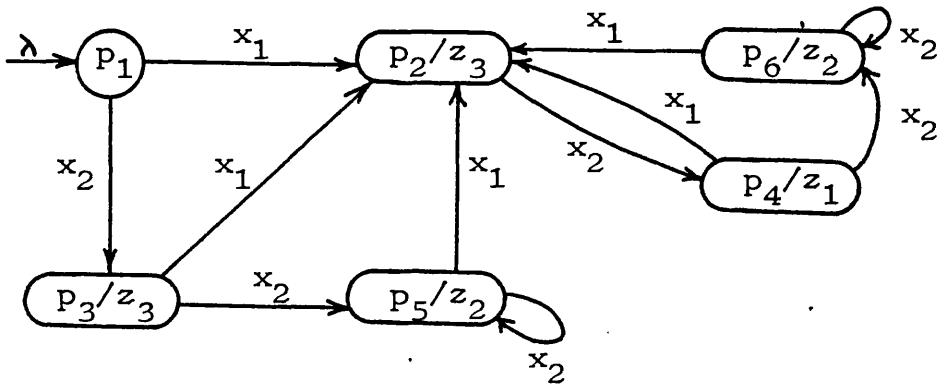
from the graph, so that each of the objective output events can be defined as a union of some of the equivalence classes. The approach is based on the work of Rabin and Scott, who consider event graphs generating single output events as "nondeterministic" automata, and prove that such an automaton defines a deterministic automaton with the same output event [Rabin & Scott]. By similar reasoning, graph 3.6 can be used to derive an automaton with Mz_1, Mz_2 and Mz_3 as output events. To derive this automaton consider the event A on graph 3.6, this being the event containing the blank tape λ. Event A has three x_1-successors C, F and Mz_3, and will be said to have "x_1-successor set" $\{C,F,Mz_3\}$, similarly event A has x_2-successor set $\{B,E,H,Mz_3\}$. These observations are expressed as row $\{A\}$ of the table and introduce two further rows $\{C,F,Mz_3\}$ and $\{B,E,H,Mz_3\}$. None of the events C,F,Mz_3 has a x_1-successor, however C has x_2-successor set $\{D\}$, and F has x_2-successor set $\{Mz_1\}$, so by forming the union the set $\{C,F,Mz_3\}$ is given the x_2-successor set $\{D,Mz_1\}$.



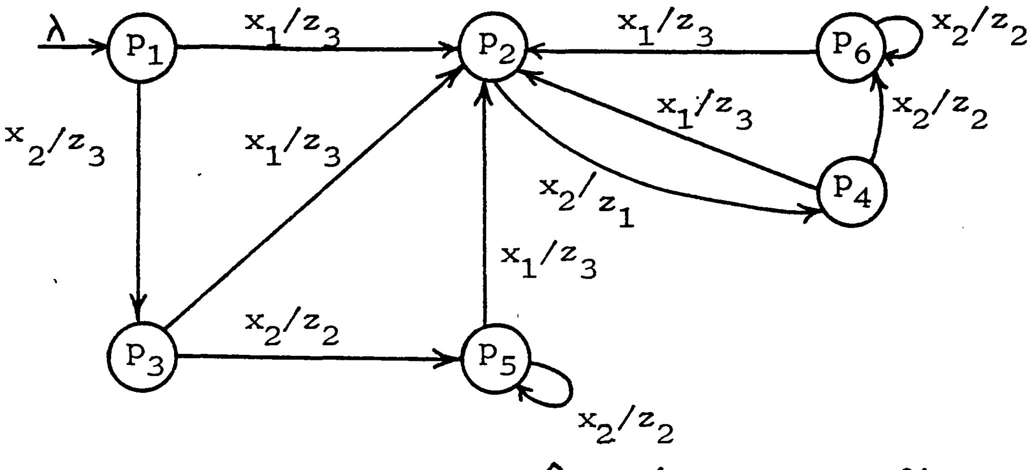
Graph (a)



Graph (b) Event semiautomaton $\langle S_P \overline{X}_P \rangle$



Graph (c) Event automaton $\hat{P} = \langle S_P X_P Z_P \overline{X}_P \omega_P \rangle$



Graph (d) Event automaton $\hat{P} = \langle S_P X_P Z_P \overline{X}_P \tilde{X}_P \rangle$

Figure 3.7

The completion of rows $\{C,F,Mz_3\}$ and $\{B,E,H,Mz_3\}$ in this way introduces further rows $\{D,Mz_1\}$ and $\{B,E,H,Mz_2\}$, and the strategy is obvious, whenever a row introduces an entry not previously encountered a further row is formed. It is not evident that the process must terminate, however this follows from the work of Rabin and Scott, and the row $\{B,E,G,Mz_2\}$ is completed without introducing further rows.

The columns x_1,x_2 of the table represent indexed mappings over $\mathcal{P}(\psi)$, where $\mathcal{P}(\psi)$ is the set of the subsets of $\psi = \{A,B,\ldots.H,Mz_1,Mz_2,Mz_3\}$, for example the mapping indexed by x_1 assigns $\{B,E,G,Mz_2\}$ to $\{C,F,Mz_3\}$. These mappings can be expressed in the form of graph 3.7(a), and the relationship between the graphs 3.6 and 3.7(a) can be considered by introducing the natural relation R, which relates each event on graph 3.6 to the subsets containing this event. For example R relates A to $\{A\}$, and relates B to $\{B,E,H,Mz_2\}$, $\{B,E,G,Mz_2\}$ and $\{B,E,H,Mz_3\}$ since B appears in each of these sets. Figure 3.8(a) illustrates that R relates B to $\{B,E,H,Mz_3\}$ and shows, in accordance with graph 3.6, that C is a x_1-successor of B.

$$\begin{tikzcd} B \arrow[r, "R"] \arrow[d, "x_1"'] & \{B,E,H,Mz_3\} \\ C & \end{tikzcd}$$

(a)

$$\begin{tikzcd} & \{B,E,H,Mz_3\} \arrow[d, "x_1"] \\ C \arrow[r, "R"] & \{C,F,Mz_3\} \end{tikzcd}$$

(b)

Figure 3.8

In defining a x_1-successor for $\{B,E,H,Mz_3\}$ on graph 3.7(a), however, the x_1-successor $\{C,F,Mz_3\}$ was formed from the x_1-successors of B,E,H, and Mz_3 on graph 3.6, therefore $\{C,F,Mz_3\}$ must contain the x_1-successor of B, that is R must relate C to $\{C,F,Mz_3\}$, as shown in figure 3.8(b). Such reasoning confirms that the relation R is a weak homomorphism of graph 3.6 onto graph 3.7(a), so graph 3.7(a) represents a weak-homomorphic image.

Each subset on graph 3.7(a) represents a regular event, and this can be investigated by forming graph 3.7(b), for example $\{A\}$ has been replaced by the event p_1. The arcs on graph 3.7(b) express that the events interrelate under concatenation, for example the x_1 arc from p_1 to p_2 expresses $p_1 \circ x_1 \subseteq p_2$, and the graph can be formalised as the event semiautomaton $\langle S_P \ \overline{X}_P \rangle$, where $S_P = \{p_1, p_2, \ldots p_6\}$, and $\overline{X}_P = \{\overline{x_1}^P, \overline{x_2}^P\}$ [redacted] is a set of mappings over S_P. In fact S_P is a partition of the valid event V, and the equivalence E associated with the partition S_P is "right invariant" over V, so that if t_a and t_b are arbitrary tapes where $t_a \equiv t_b$ (E), and x is an input from X, then either $t_a \circ x, t_b \circ x \not\in V$ or $t_a \circ x \equiv t_b \circ x$ (E). Furthermore each output event can be expressed as a union of certain equivalence classes of the partition $S_P = \{p_1, p_2, \ldots p_6\}$, the equivalence classes being deduced from graph 3.7(a). Specifically $\{D,Mz_1\}$ was replaced by p_4, so $Mz_1 = p_4$, similarly $\{B,E,H,Mz_2\}$ was replaced by

p_5 and $\{B,E,G,Mz_2\}$ by p_6, giving $Mz_2 = p_5 \cup p_6$, and finally $Mz_3 = p_2 \cup p_3$. These observations are expressed in graph 3.7(c), for example $Mz_2 = p_5 \cup p_6$ is expressed by associating output symbol z_2 with the "state events" p_5 and p_6.

This approach is based on the work of Rabin and Scott, however it is particularly important to recognise that the resulting graph 3.7(c) represents an "event automaton", and can be formalised as $\hat{P} = \langle S_P\ X_P\ Z_P\ \overline{X}_P\ \omega_P \rangle$ where $S_P = \{p_1 \dots p_6\}$ is a partition of the valid event V, $X_P = \{x_1, x_2\}$, $Z_P = \{z_1, z_2, z_3\}$, $\overline{X}_P = \{\overline{x_1}^P, \overline{x_2}^P\}$ and $\omega_P = \{\langle p_2\ z_3\rangle\ \langle p_3\ z_3\rangle\ \langle p_4\ z_1\rangle\ \langle p_5\ z_2\rangle\ \langle p_6\ z_2\rangle\}$.
This is a crucially important concept since the idea of a "state" has been discarded, instead the event automaton represents a system of interrelated regular events and shows that each objective output event can be expressed as a union of certain "state events". Specifically $Mz_1 = p_4$, $Mz_2 = p_5 \cup p_6$ and $Mz_3 = p_2 \cup p_3$, where the state events on the graph form a right-invariant equivalence over V, so the event automaton shows that each objective output event can be expressed as a union of certain equivalence classes.

Furthermore, the same relationships can be expressed in Mealy form. For example the x_1-successor of $\{A\}$ on graph 3.7(a) is $\{C,F,Mz_3\}$, and Mz_3 appears within this set so z_3 can be associated with the arc from p_1 to p_2. Continuing this reasoning gives the graph of figure 3.7(d), and the graph can be formalised as the event automaton $\hat{P} = \langle S_P\ X_P\ Z_P\ \overline{X}_P\ \widetilde{X}_P \rangle$ where $\widetilde{X}_P = \{ \widetilde{x}_1^P, \widetilde{x}_2^P \}$ is a set of mappings from the partition S_P to Z_P, specifically

$\widetilde{x}_1^P = \{ \langle p_1\ z_3 \rangle \langle p_3\ z_3 \rangle \langle p_4\ z_3 \rangle \langle p_5\ z_3 \rangle \langle p_6\ z_3 \rangle \}$ and

$\widetilde{x}_2^P = \{ \langle p_1\ z_3 \rangle\ \langle p_2\ z_1 \rangle\ \langle p_3\ z_2 \rangle\ \langle p_4\ z_2 \rangle\ \langle p_5\ z_2 \rangle\ \langle p_6\ z_2 \rangle \}$.

In this case each output event is defined as a union of "X-concatenated" state events,

$$Mz_1 = p_2 \circ x_2,$$

$$Mz_2 = p_3 \circ x_2 \cup p_4 \circ x_2 \cup p_5 \circ x_2 \cup p_6 \circ x_2,$$

and $$Mz_3 = p_1 \circ x_1 \cup p_1 \circ x_2 \cup p_3 \circ x_1 \cup p_4 \circ x_1 \cup p_5 \circ x_1 \cup p_6 \circ x_1$$

These event automata can be regarded as "finite-state expressions" of the objective mapping $\Gamma obj : X^* \longrightarrow Z$, in the Moore and Mealy forms. It is important to observe, however, that a finite-state expression of a mapping is never unique, for example graph 3.7(c) is not the only way of expressing Γobj in Moore form, and there might be an alternative representation with fewer state events. It remains to process the event automata to give more elegant finite-state expressions of Γobj, in fact Γobj can be expressed using just three state events. That is, there exists a right-invariant equivalence over V with just three equivalence classes, such that each of the objective output events can be defined as an appropriate union. The resulting event automaton can then be finalised to give a basis for designing a sequential circuit.

3.3 Formalising the objective automaton

Before further consideration is given to the event automaton of graph 3.7(c), it is important to note that no output symbol has been associated with the event p_1. This is because λ was excluded from the objective output events Mz_1, Mz_2 and Mz_3, so these events formed a partition of $V - \lambda$, rather than a partition of the valid event V. Care must be taken, however, to ensure that neither of the output symbols z_1, z_2 are associated with the valid event p_1, and graph 3.7(c) should be adjusted so that the neutral output symbol z_3 is associated with the event p_1, which then becomes p_1/z_3. This adjustment could have been avoided by including λ in the output event Mz_3 from the outset, however this would have committed the design to the Moore form, and it would not have been possible to express this objective as a Mealy automaton.

The event automaton of graph 3.7(c) represents a system of interrelated regular events, and the way the state-events are related can be expressed as the identities

$$p_1 = \lambda$$

$$p_2 = p_1 \circ x_1 \cup p_3 \circ x_1 \cup p_4 \circ x_1 \cup p_5 \circ x_1 \cup p_6 \circ x_1$$

$$p_3 = p_1 \circ x_2$$

$$p_4 = p_2 \circ x_2$$

$$p_5 = p_3 \circ x_2 \cup p_5 \circ x_2$$

$$p_6 = p_4 \circ x_2 \cup p_6 \circ x_2$$

Furthermore, having made the above adjustment, the output events represented on the graph can be defined by the identities

$$Pz_1 = p_4$$

$$Pz_2 = p_5 \cup p_6$$

$$Pz_3 = p_1 \cup p_2 \cup p_3$$

Since $Pz_2 = p_5 \cup p_6$, define $p_{56} = p_5 \cup p_6$. Then output event Pz_2 can be defined using just one state event, that is $Pz_2 = P_{56}$, and from above $p_5 = p_3 \circ x_2 \cup p_5 \circ x_2$ and $p_6 = p_4 \circ x_2 \cup p_6 \circ x_2$ so $p_{56} = p_3 \circ x_2 \cup p_4 \circ x_2 \cup (p_5 \cup p_6) \circ x_2$, giving $p_{56} = p_3 \circ x_2 \cup p_4 \circ x_2 \cup p_{56} \circ x_2$. Furthermore the union $p_5 \cup p_6 = p_{56}$ can be formed throughout the system of identities, in particular

$p_2 = p_1 \circ x_1 \cup p_3 \circ x_1 \cup p_4 \circ x_1 \cup (p_5 \cup p_6) \circ x_1$, so the system of identies can be rewritten

$$p_1 = \lambda$$

$$p_2 = p_1 \circ x_1 \cup p_3 \circ x_1 \cup p_4 \circ x_1 \cup p_{56} \circ x_1$$

$$p_3 = p_1 \circ x_2$$

$$p_4 = p_2 \circ x_2$$

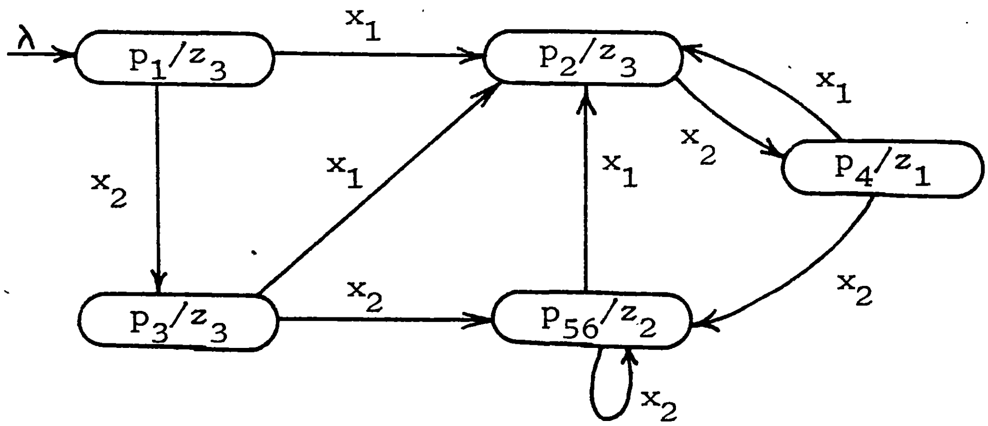
$$p_{56} = p_3 \circ x_2 \cup p_4 \circ x_2 \cup p_{56} \circ x_2$$

$$Pz_1 = p_4$$

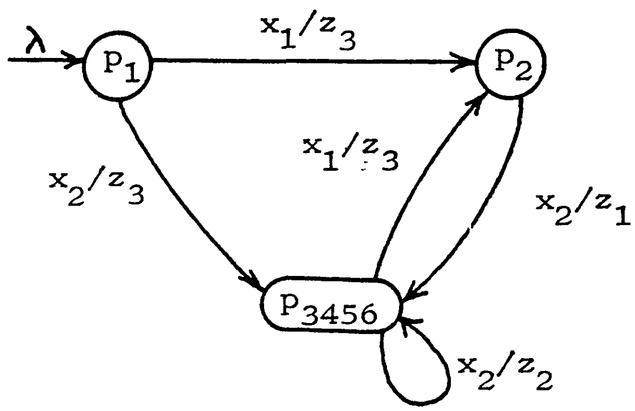
$$Pz_2 = p_{56}$$

$$Pz_3 = p_1 \cup p_2 \cup p_3$$

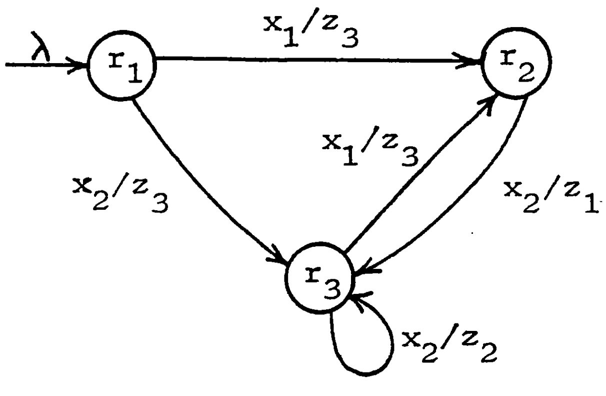
These identities define the event automaton of graph 3.9(a), which defines the output events Pz_1, Pz_2 and Pz_3 as before, but involves five state events instead of six. The analysis has determined a right-invariant equivalence over V with five equivalence classes, such that the output events Pz_1, Pz_2 and Pz_3 can be defined as appropriate unions. This



Graph (a)



Graph (b)



Graph (c) Event automaton $\hat{R} = \langle S_R\ X_R\ Z_R\ \bar{X}_R\ \tilde{X}_R \rangle$

Figure 3.9

observation is closely related to automaton reduction using state-equivalence, the equivalence partition of the automaton of graph 3.7(c) being $(p_1)(p_2)(p_3)(p_4)(p_5p_6)$.

Similarly, the Mealy event automaton of graph 3.7(d) can be reduced by considering the identities

$p_1 = \lambda$

$p_2 = p_1 \circ x_1 \cup p_3 \circ x_1 \cup p_4 \circ x_1 \cup p_5 \circ x_1 \cup p_6 \circ x_1$

$p_3 = p_1 \circ x_2$

$p_4 = p_2 \circ x_2$

$p_5 = p_3 \circ x_2 \cup p_5 \circ x_2$

$p_6 = p_4 \circ x_2 \cup p_6 \circ x_2$

$Pz_1 = p_2 \circ x_2$

$Pz_2 = p_3 \circ x_2 \cup p_4 \circ x_2 \cup p_5 \circ x_2 \cup p_6 \circ x_2$

$Pz_3 = p_1 \circ x_1 \cup p_1 \circ x_2 \cup p_3 \circ x_1 \cup p_4 \circ x_1 \cup p_5 \circ x_1 \cup p_6 \circ x_1$.

Since $\underline{Pz_3 = p_1 \circ x_1 \cup p_1 \circ x_2 \cup (p_3 \cup p_4 \cup p_5 \cup p_6) \circ x_1}$

define

$p_{3456} = p_3 \cup p_4 \cup p_5 \cup p_6$. Then from above

$p_3 = p_1 \circ x_2$, $p_4 = p_2 \circ x_2$, $p_5 = p_3 \circ x_2 \cup p_5 \circ x_2$ and

$p_6 = p_4 \circ x_2 \cup p_6 \circ x_2$ so

$p_{3456} = p_1 \circ x_2 \cup p_2 \circ x_2 \cup (p_3 \cup p_4 \cup p_5 \cup p_6) \circ x_2$, giving

$p_{3456} = p_1 \circ x_2 \cup p_2 \circ x_2 \cup p_{3456} \circ x_2$.

Then the system of identities can be rewritten

$p_1 = \lambda$

$p_2 = p_1 \circ x_1 \cup p_{3456} \circ x_1$

$p_{3456} = p_1 \circ x_2 \cup p_2 \circ x_2 \cup p_{3456} \circ x_2$

$Pz_1 = p_2 \circ x_2$

$Pz_2 = p_{3456} \circ x_2$

$Pz_3 = p_1 \circ x_1 \cup p_1 \circ x_2 \cup p_{3456} \circ x_1$

These identities define the Mealy event automaton of figure 3.9(b), and this is a particularly elegant event automaton since the output events Pz_1, Pz_2 and Pz_3 are defined using just three equivalence classes.

The event automaton of graph 3.9(b) represents a right-invariant equivalence over V where the output events Pz_1, Pz_2 and Pz_3 are expressed as a union of X-concatenated equivalence classes, for example
$Pz_3 = p_1 \circ x_1 \cup p_1 \circ x_2 \cup p_{3456} \circ x_1$. This suggests a "Mealy variation" of the Nerode theorem, and to investigate the event automaton further let events r_1, r_2 and r_3 replace the events p_1, p_2 and p_{3456}, to give the automaton $\hat{R} = \langle S_R\ X_R\ Z_R\ \overline{X}_R\ \tilde{X}_R \rangle$ of figure 3.9(c). Then

$$r_1 = \lambda \qquad \text{(i)}$$

$$r_2 = r_1 \circ x_1 \cup r_3 \circ x_1 \qquad \text{(ii)}$$

$$r_3 = r_1 \circ x_2 \cup r_2 \circ x_2 \cup r_3 \circ x_2 \qquad \text{(iii)}$$

and the output events represented by the automaton $\hat{R}$ can be defined as

$Rz_1 = r_2 \circ x_2$

$Rz_2 = r_3 \circ x_2$

and $Rz_3 = r_1 \circ x_1 \cup r_1 \circ x_2 \cup r_3 \circ x_1$

Since $r_1 = \lambda$ the identity (iii) becomes

$r_3 = r_3 \circ x_2 \cup (x_2 \cup r_2 \circ x_2)$, and this is of the form $X = X \circ A \cup B$ where $\lambda \notin A$, in which case the solution is $X = B \circ A^*$ [Arden]. Therefore

$r_3 = (x_2 \cup r_2 \circ x_2) \circ x_2^*$, and then (ii) becomes

$r_2 = r_2 \circ x_2 \circ x_2^* \circ x_1 \cup (x_1 \cup x_2 \circ x_2^* \circ x_1)$,

again of the form $X = X \circ A \cup B$ where $\lambda \notin A$, giving

$$r_2 = (x_1 \cup x_2 \circ x_2^* \circ x_1) \circ (x_2 \circ x_2^* \circ x_1)^*.$$

Finally this solution for r_2 can be substituted in the above identity $r_3 = (x_2 \cup r_2 \circ x_2) \circ x_2^*$, and the solutions can be written

$r_1 = \lambda$

$r_2 = x_1(x_2x_2^*x_1)^* \cup (x_2x_2^*x_1)(x_2x_2^*x_1)^*$

$r_3 = x_1(x_2x_2^*x_1)^*x_2x_2^* \cup x_2x_2^* \cup (x_2x_2^*x_1)(x_2x_2^*x_1)^*x_2x_2^*.$

For convenience the concatenation operator has been omitted throughout, and will subsequently be implicit.

Consider now the set of all the valid tapes with x_1 as the first and also the last symbol. Trivially the tape x_1 has this property, and this is obviously the only such tape with just one occurrence of the input symbol x_1. Furthermore any of these tapes involving just two occurrences of the symbol x_1 must belong to the event defined by the expression $x_1x_2x_2^*x_1$, since the expression defines the set of all the valid tapes having x_1 as the first and also the last symbol. The phrase $x_2x_2{}^*$ ensures that there is at least one intervening symbol x_2, since adjacent occurrences of x_1 are invalid. Similarly the regular expression $x_1x_2x_2{}^* x_1 x_2x_2{}^* x_1$ defines the set of all these tapes with three occurrences of x_1, and it is evident that $x_1(x_2x_2{}^*x_1)^*$ defines the set of all the valid tapes with x_1 as the first and also the last symbol, since $x_1(x_2x_2{}^*x_1)^*$ can be written

$x_1 \cup x_1x_2x_2{}^*x_1 \cup x_1x_2x_2{}^*x_1x_2x_2{}^*x_1 \cup \ldots.$

Similarly $x_2x_2{}^*x_1(x_2x_2{}^*x_1)^*$ defines the set of all the valid tapes with x_2 as the first symbol and x_1 as the last, in which case the union of these events, that is

$$r_2 = x_1(x_2x_2{}^*x_1)^* \cup x_2x_2{}^*x_1(x_2x_2{}^*x_1)^*$$

consists of all the valid tapes ending with the symbol x_1. By similar reasoning, the above event r_3 consists of all the valid tapes ending with the symbol x_2.

This gives an enhanced appreciation of the event automaton $\hat{R}$ of figure 3.9(c). The state events are r_1, r_2 and r_3, where the event r_1 consists of the blank tape λ, event r_2 consists of the valid tapes ending with the symbol x_1 and event r_3 consists of the valid tapes ending x_2. Then the mappings $\overline{x_1}^R$ and $\overline{x_2}^R$, represented on the graph by the labelled arcs, express the way these events are related under concatenation. For example the x_2-arc from event r_2 to event r_3 expresses $r_2 \circ x_2 \subseteq r_3$, that is any tape ending with the symbol x_1 becomes a tape from r_3 when concatenated by x_2, and similarly the x_1-arc from r_1 to r_2 expresses $\lambda \circ x_1 \subseteq r_2$, that is $x_1 \in r_2$. Considering now the output event $Rz_1 = r_2 \circ x_2$, then Rz_1 consists of all the valid tapes ending with the sequence $\langle x_1 x_2 \rangle$, however this is the objective output event Mz_1 so $Rz_1 = Mz_1$. Similarly $Rz_2 = r_3 \circ x_2$, so Rz_2 consists of all the valid tapes ending with the sequence $\langle x_2 x_2 \rangle$, in which case $Rz_2 = Mz_2$, and finally $Rz_3 = Mz_3$.

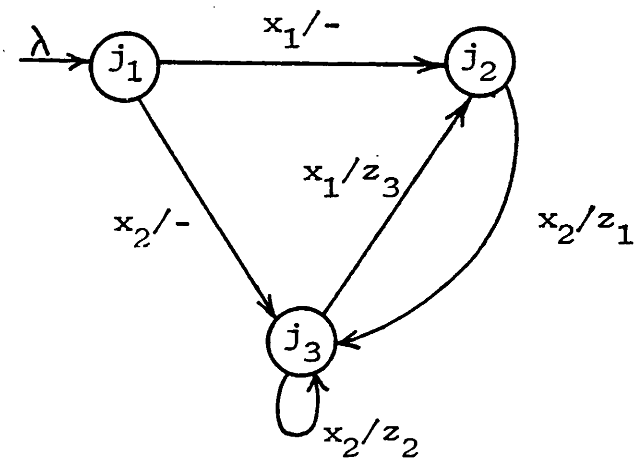
Further analysis of event automaton $\hat{R}$ can proceed as normal, for example a mapping $\bar{t}^R$ over partition S_R and a mapping $\tilde{t}^R$ from S_R to Z_R can be associated with each tape t from $X_R{}^*$. Then a mapping Σr_i from $X_R{}^*$ to S_R, where $\langle t\ r_j \rangle \in \Sigma r_i$ iff $t \in X_R{}^*$, $r_j \in S_R$ and $\langle r_i\ r_j \rangle \in \bar{t}^R$, can be associated with each of the state events r_i, and since $\lambda \in r_1$ the mapping Σr_1 associated with this state event is of particular interest. The equivalence classes defined by this mapping are the state events themselves, for example the equivalence class associated with r_2 by Σr_1 consists of all the tapes associating final successor r_2 with r_1, and this is just the event r_2. Furthermore, the equivalence classes defined by the mapping $\Gamma r_1 : X_R{}^* \longrightarrow Z_R$, where

$$\langle t\ z \rangle \in \Gamma r_1 \quad \text{iff} \quad t \in X_R{}^*,\ z \in Z_R,\ \text{and}\ \langle r_1\ z \rangle \in \tilde{t}^R,$$

can be deduced directly from the graph, and are given by $Rz_1 = r_2 \circ x_2$, $Rz_2 = r_3 \circ x_2$ and $Rz_3 = r_1 \circ x_1 \cup r_1 \circ x_2 \cup r_3 \circ x_1$, where from above $Rz_1 = Mz_1$, $Rz_2 = Mz_2$ and $Rz_3 = Mz_3$. Consequently the mappings Γr_1 and Γobj define identical equivalence classes, in which case $\Gamma r_1 = \Gamma obj$, confirming that the event automaton $\hat{R}$ is a "finite-state expression" of the objective mapping.

Consequently, the automaton $\hat{R}$ provides a basis for designing a sequential circuit with the desired behaviour. Before formalising the circuit design, however, it should be considered whether the output environment is "intermittent". Consider, for example, a block decoder, that is a circuit designed to associate an appropriate output code with input sequences of a fixed length. For example it might be desired to associate a specific output code with sequences of four input codes where a particular input code appears twice or more. Such a sequential circuit will produce a significant output code for every fourth input code, and it might be advantageous to design the output environment so that intermediate output codes are disregarded. In such cases the output environment can be considered intermittent, and the ignored output codes can be arbitrary.

In fact the example design objective involves such intermittency, and this can be appreciated by considering the neglected part (c) of the objective statement, the "properties of the output environment". Here it is given that the output environment is such that any output symbol can be associated with tapes of unit length, since presumably the output environment will disregard the very first output code produced by the sequential system. Then the output code associated with either of the unit length tapes is irrelevant, and the symbol z_3 can be removed from the x_1-arc relating r_1 to r_2, and from the x_2-arc relating r_1 to r_3. The event automaton $\hat{J} = \langle S_J\ X_J\ Z_J\ \overline{X}_J\ \tilde{X}_J \rangle$ of figure 3.10, the "objective automaton", is defined accordingly, and the mapping Γj_1 associated with the event j_1 provides an exact expression of the design objective.



<u>Objective automaton $\hat{J} = \langle S_J\ X_J\ Z_J\ \overline{X}_J\ \tilde{X}_J \rangle$</u>

<u>Figure 3.10</u>

<u>3.4 Conclusion</u>

The preceding shows that an "event automaton" is an automaton with state events instead of states, and shows that event automata are closely related to the Nerode theorem. A given automaton $\hat{A} = \langle S_A\ X_A\ Z_A\ \overline{X}_A\ \omega_A \rangle$, with a given reference state $a_i \in S_A$, can be converted to the corresponding event automaton $\hat{E} = \langle S_E\ X_E\ Z_E\ \overline{X}_E\ \omega_E \rangle$ by defining $X_E = X_A$, $Z_E = Z_A$, and defining S_E to be the set of the equivalence classes associated with the states from S_A. Then S_E is the set of subsets of X_A^* called "state events", where the tapes forming a state event relate a common final-successor from S_A with the reference state a_i, and if automaton $\hat{A}$ is a_i-connected the partition S_E will be in one-to-one correspondence with S_A. The event semiautomaton $\langle S_E \; \overline{X}_E \rangle$ will express the way the blocks of partition S_E interrelate under concatenation, and the event semiautomaton will be closely related to semiautomaton $\langle S_A \; \overline{X}_A \rangle$ since an association $\langle a_j \; a_k \rangle \in \bar{x}^A$ implies $E_j \circ x \subseteq E_k$ and then $\langle E_j \; E_k \rangle \in \bar{x}^E$, where E_j and E_k are the state events from S_E associated with the states a_j and a_k from S_A. Furthermore the mapping ω_E from S_E to Z_E expresses the way each output event is a composition of certain state events. For example an association $\langle E \; z_p \rangle \in \omega_E$ expresses that the state event E from S_E is a subset of the output event M_p, where M_p is the set of all the tapes associating final-output z_p with reference state a_i.

Such an event automaton provides a clear visualisation of the Nerode theorem extended to multiple outputs, however the Nerode theorem also provides an approach to sequential circuit synthesis, so event automata are important in expressing the design aim. The design process begins by recognising the set of input tapes to be associated with each output symbol, then the aim is to determine a right-invariant equivalence over the valid event so that each of these "objective" events can be expressed as a union of certain equivalence classes. Such a right-invariant equivalence can be formalised as an "objective" event automaton, and in general this automaton will be partial since the state events will include only valid tapes.

This is the idea behind the "intuitive" approach, although the intuitive designer will often form an objective event automaton without realising that the graph nodes represent events, and without appreciating that the approach is based on the Nerode theorem. The formal procedures have this same basic aim, and consideration has been given to the "regular event" graphs originated by Ott & Feinstein. It has been shown that each node on such a graph represents a regular event, and one event can be a subset of several others so the graph represents a right-invariant cover of valid event V, rather than a right-invariant partition. This interpretation can be used to replace the idea of a "nondeterministic" automaton [Rabin & Scott; Kohavi], furthermore an arc labelled λ from an event E to an event E' on the graph expresses $E \circ \lambda \subseteq E'$, that is $E \subseteq E'$, and this replaces the idea of "instantaneous" transitions.

A regular-event graph can be constructed for any "restricted" regular event, that is for any regular event defined using the union, concatenation and iteration operators only. In general the regular expressions defining the objective output events will involve intersection and negation, in which case an alternative graphical method [McNaughton & Yamada] or an approach using derivatives [Brzozowski] can be adopted. By the "derivative" $D_s E$ of a regular event E with respect to a tape s is meant the set of all the tapes t so that $s \circ t \in E$, that is $D_s E$ consists of all the final subtapes of the tapes from E with initial subtape s. The distinct derivatives of a regular event E can be adopted as the state events of an event automaton, and the event E will be a union of certain state events. Certainly this approach has advantages over the others, however it is particularly important to appreciate that the various procedures relate to a common problem, of finding a right-invariant equivalence in accordance with the Nerode theorem.

CHAPTER FOUR : Automaton Realisation using standard sequential units

4.1 Introduction

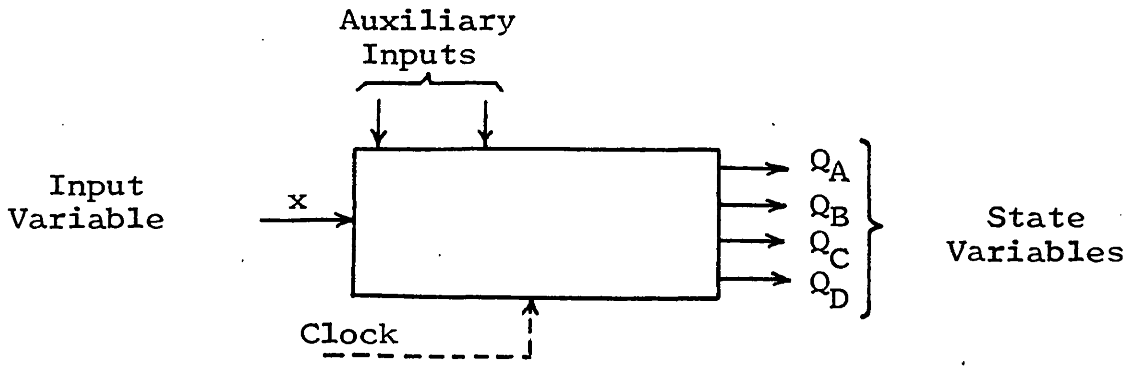
Once the design aim is formalised as an objective automaton, the designer can consider the translation of this automaton into a hardware realisation. The translation process discussed in various texts [Lewin; Miller] begins with the selection of a bistable type (JK, SR or D type), and produces a realisation of the objective automaton as an interconnection of bistables and combinatorial elements.

However various sequential systems, such as counters and registers, are available in MSI (Medium Scale Integration) form, and the use of these units offers important advantages. Circuits incorporating MSI units can be more compact, more reliable and more economical than their counterparts using bistables, in addition they are easier to test and are more easily assembled. Consequently the designer is motivated to realise the objective automaton using MSI units, but the standard design process is inapplicable, since in using this approach the use of bistables is accepted from the outset. As no alternative approach has been developed, the designer will usually rely on intuition.

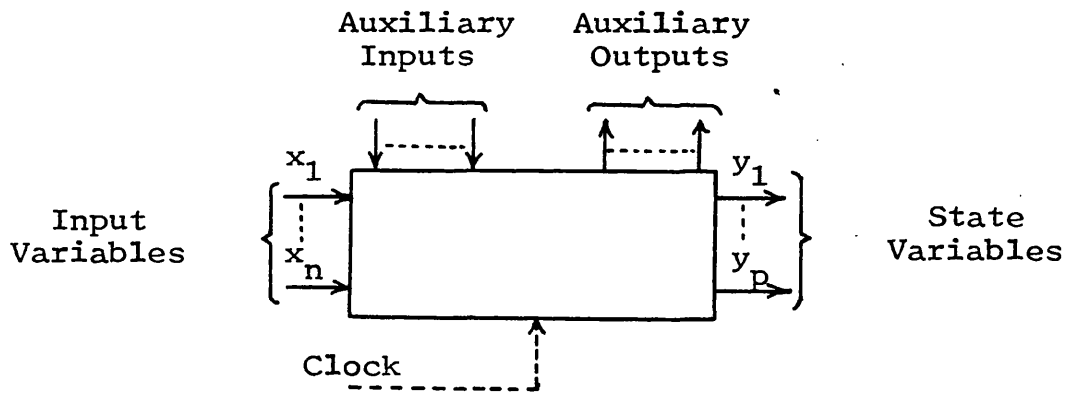
Automaton realisation using standard sequential units is considered in the present chapter, and the aim is to investigate the way a standard sequential unit can be assessed as a basis for realising a given objective automaton. The example objective automaton will be the automaton $\hat{J}$ derived previously, however the "event automaton" concept can often be suppressed, and the state events j_1, j_2 and j_3 will be referred to as "states" of the objective automaton. The mapping $\Gamma_{j_1} : X_J^* \longrightarrow Z_J$ associated with objective state j_1 will be of particular interest, since this expresses the objective translation from input sequences to outputs, and automaton $\hat{J}$ can be regarded as a "finite-state" representation of this objective translation.

Before considering the realisation of automaton $\hat{J}$ using standard sequential units, however, it is important to consider the way the standard units can be represented. As an example, figure 4.1 (a) represents a four-stage shift register in MSI form. The shift register has an input variable x, has a "clock" input to synchronise the state transitions and has four state variables Q_A, Q_B, Q_C and Q_D, these being the outputs of the four bistables forming the register. Some MSI shift registers are quite complex, for example it might be possible to preset the shift register to an arbitrary initial configuration, and the controlling inputs are shown in figure 4.1(a) as "auxiliary" inputs.

More generally, a standard sequential system or "stock unit" will take the form of figure 4.1(b).



(a) MSI Shift Register



(b) Arbitrary Stock Unit

Figure 4.1

Here $x_1, x_2, \ldots . x_n$ are the input variables and $y_1, y_2, \ldots . y_p$ are the state variables. The auxiliary input variables (such as CLEAR, PRESET, LOAD, ENABLE) control the operation mode of the unit, and a number of auxiliary output variables (such as CARRY) relate to the usual application of the unit. For present purposes it will be assumed that the auxiliary inputs are set to a specific configuration, so that the stock unit operates in a specific mode, and interest will be confined to the state variables, the auxiliary outputs being disregarded.

Then the input variables define a set X of input codes, the state variables define a set S of state codes, and changes of state in response to applied inputs can be expressed as a semiautomaton $\langle S\ \overline{X} \rangle$. Hence each stock unit can be represented as a unary algebra, and an interconnection of MSI units can be represented as a system of interactive unary algebras. Subsequently (Chapter 5) the basic interconnection schemes for unary algebras will be considered, and each unary algebra can be taken to represent a MSI unit, so the study will relate directly to systems of interdependent sequential units. Then the problem of realising an objective automaton in composite form, as an interconnection of standard units, can be considered (Chapter 6). To begin, however, it is assumed that a specific "stock" unit is available, and the problem will be that of assessing the value of this stock unit as a direct realisation of the objective automaton. It may be that a given objective automaton and a given stock unit are dissimilar, in which case the stock unit cannot be directly useful in realising the objective automaton. The main problem is to establish the meaning of "similarity" between sequential systems, and to clarify what is meant by a "realisation".

4.2 Automaton realisation using stock units

The approach using stock units will be illustrated with the example objective automaton $\hat{J} = \langle S_J\ X_J\ Z_J\ \overline{X}_J\ \tilde{X}_J \rangle$ from Chapter 3, and for convenience the automaton $\hat{J}$ and the objective semiautomaton $J = \langle S_J\ \overline{X}_J \rangle$ are repeated as the tables (a) and (b) of figure 4.2. It will then be assumed that the stock unit of figure 4.2(c) is available, where the state transitions of the stock unit are known from the data sheets and are expressed as table 4.2(d). To begin it is necessary to formalise the stock unit as a semiautomaton, so an input set $X_C = \{0,1\}$ and a state set $S_C = \{ \langle 00 \rangle \langle 01 \rangle \langle 10 \rangle \langle 11 \rangle \}$ are defined. Then table 4.2(d) shows that state code $\langle 00 \rangle$ becomes $\langle 01 \rangle$ for input 0, $\langle 01 \rangle$ becomes $\langle 11 \rangle$, $\langle 10 \rangle$ becomes $\langle 00 \rangle$ and $\langle 11 \rangle$ becomes $\langle 00 \rangle$, and these associations can be expressed as a mapping $\overline{0}^C$ over S_C where

$$\overline{0}^C = \{[\langle 00 \rangle\ \langle 01 \rangle][\langle 01 \rangle\ \langle 11 \rangle][\langle 10 \rangle\ \langle 00 \rangle][\langle 11 \rangle\ \langle 00 \rangle]\}.$$

Similarly the 1-column of table 4.2(d) represents a mapping $\overline{1}^C$ over S_C, where

$$\overline{1}^C = \{[\langle 00 \rangle\ \langle 11 \rangle][\langle 01 \rangle\ \langle 00 \rangle][\langle 10 \rangle\ \langle 11 \rangle][\langle 11 \rangle\ \langle 11 \rangle]\}.$$

Defining $\overline{X}_C = \{\overline{0}^C, \overline{1}^C\}$, the stock unit can then be represented as the stock semiautomaton $C = \langle S_C\ \overline{X}_C \rangle$, where by a "stock semiautomaton" is meant any semiautomaton representing the state transitions of a stock unit.

The problem now is to relate objective semiautomaton $J = \langle S_J\ \overline{X}_J \rangle$, as shown in figure 4.2(e), to the stock semiautomaton $C = \langle S_C\ \overline{X}_C \rangle$ represented in figure 4.2(f).

	x_1	x_2
j_1	j_2/-	j_3/-
j_2	-/-	j_3/z_1
j_3	j_2/z_3	j_3/z_2

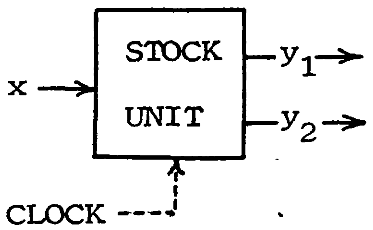
(a) Objective automaton

$\hat{J} = \langle S_J\ X_J\ Z_J\ \overline{X}_J\ \tilde{X}_J \rangle$

	x_1	x_2
j_1	j_2	j_3
j_2	-	j_3
j_3	j_2	j_3

(b) Objective semiautomaton

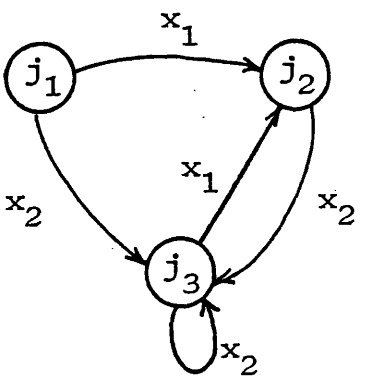
$J = \langle S_J\ \overline{X}_J \rangle$



(c) Stock Unit

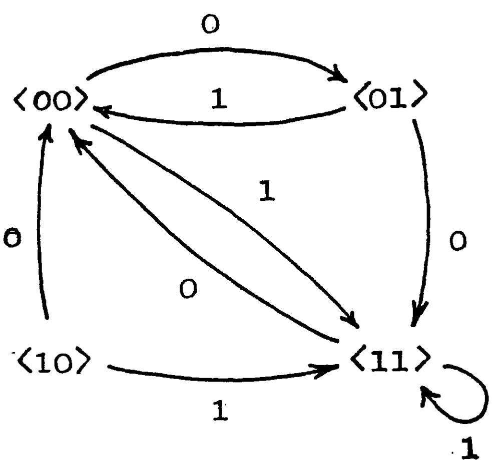
		x	
		0	1
$\langle y_1 y_2 \rangle$	$\langle 00 \rangle$	$\langle 01 \rangle$	$\langle 11 \rangle$
	$\langle 01 \rangle$	$\langle 11 \rangle$	$\langle 00 \rangle$
	$\langle 10 \rangle$	$\langle 00 \rangle$	$\langle 11 \rangle$
	$\langle 11 \rangle$	$\langle 00 \rangle$	$\langle 11 \rangle$

(d) State transitions of stock unit



(e) Objective semiautomaton

$J = \langle S_J\ \overline{X}_J \rangle$



(f) Stock semiautomaton

$C = \langle S_C\ \overline{X}_C \rangle$

Figure 4.2

Firstly each input symbol from X_J must be assigned to a distinct code in X_C, and this can be achieved by defining an arbitrary injection of X_J into X_C, for example define $\alpha = \{\langle x_1\ 0\rangle \langle x_2\ 1\rangle\}$. Then input assignment α assigns input symbol x_1 to code 0, which will be written $x_1)\alpha = 0$, and symbol x_2 is assigned to code 1 so $x_2)\alpha = 1$.

Similarly each state symbol from state set S_J must be assigned to a distinct code in S_C, however this state assignment must be carefully considered. Firstly the assignment γ from S_J to S_C must have domain $D[\gamma] = S_J$, to ensure that each objective state is allocated to a code, but this assignment can be one-many in nature, rather than one-one. Thus an objective state can be allocated more than one code, but once a code is allocated to one objective state it cannot be allocated to another. Furthermore the state assignment γ must be a weak homomorphism under the input assignment α, that is γ must be defined so that
$(\forall x)(x \in X_J \implies \gamma^{-1}\ \bar{x}^J \subseteq \overline{x)\alpha}^C\ \ \gamma^{-1})$.

Formal justification for this requirement will be given later, but the basic idea is that of weak homomorphism with a transposition of input symbols, and is illustrated in figure 4.3(a). Here the state assignment γ allocates a code c from S_C to a state j from S_J, and j' is the $\bar{x}^J$-successor of state j, that is $\langle c\ j\rangle \in \gamma^{-1}$ and $\langle j\ j'\rangle \in \bar{x}^J$, in which case $\langle c\ j'\rangle \in \gamma^{-1}\bar{x}^J$.

$$
\begin{tikzcd}
j \in S_J \arrow[d, "\bar{x}^J"'] & & c \in S_C \arrow[ll, "\gamma^{-1}"'] \\
j' \in S_J & &
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & c \arrow[d, "\overline{x)\alpha}^C"] \\
j' & c' \in S_C \arrow[l, "\gamma^{-1}"']
\end{tikzcd}
$$

(a) $\langle c\ j' \rangle \in \gamma^{-1}\bar{x}^J$ implies $\langle c\ j' \rangle \in \overline{x)\alpha}^C\ \gamma^{-1}$

$$
\begin{tikzcd}
j_1 \arrow[d, "\bar{x}_1^J"'] & & \langle 00 \rangle \arrow[ll, "\gamma^{-1}"'] \\
j_2 & &
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & \langle 00 \rangle \arrow[d, "\bar{0}^C"] \\
j_2 & \langle 01 \rangle \arrow[l, "\gamma^{-1}"']
\end{tikzcd}
$$

(b) $[\langle 00 \rangle\ j_2] \in \gamma^{-1}\bar{x}_1^J$ implies $[\langle 00 \rangle\ j_2] \in \bar{0}^C\ \gamma^{-1}$

$$
\begin{tikzcd}
j_1 \arrow[d, "\bar{x}_2^J"'] & & \langle 00 \rangle \arrow[ll, "\gamma^{-1}"'] \\
j_3 & &
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & \langle 00 \rangle \arrow[d, "\bar{1}^C"] \\
j_3 & \langle 11 \rangle \arrow[l, "\gamma^{-1}"']
\end{tikzcd}
$$

(c) $[\langle 00 \rangle\ j_3] \in \gamma^{-1}\bar{x}_2^J$ implies $[\langle 00 \rangle\ j_3] \in \bar{1}^C\ \gamma^{-1}$

Figure 4.3

In addition input symbol x from X_J is assigned to input code $x)\alpha \in X_C$ by the input assignment α, and $x)\alpha$ is associated with a mapping $\overline{x)\alpha}^C$ over S_C since $C = \langle S_C\ \overline{X}_C \rangle$ is a X_C-semiautomaton over S_C. Consequently $\overline{x)\alpha}^C\ \gamma^{-1}$ must assign c to j', in which case γ must assign j' to the code c' where c' is the $\overline{x)\alpha}^C$-successor of code c. In effect weak homomorphism γ assigns objective state j to code c, and j' is the successor of j, so j' must be assigned to the successor of code c.

The present aim is to determine a one-many weak homomorphism, under input assignment $\alpha = \{\langle x_1\ 0\rangle\ \langle x_2\ 1\rangle\}$, of objective semiautomaton $J = \langle S_J\ \overline{X}_J\rangle$ to the stock semiautomaton $C = \langle S_C\ \overline{X}_C\rangle$, and the approach is to assume that γ is a one-many weak homomorphism relating objective state j_1 to code $\langle 00\rangle$, so $[\langle 00\rangle\ \ j_1] \in \gamma^{-1}$. From table 4.2(b) the $\overline{x_1}^J$-successor of j_1 is j_2, that is $\langle j_1\ j_2\rangle \in \overline{x_1}^J$, as shown in figure 4.3(b). Then $[\langle 00\rangle\ \ j_2] \in \gamma^{-1}\ \overline{x_1}^J$, and by assumption γ is a weak homomorphism of J to C under α so $\gamma^{-1}\ \overline{x_1}^J \subseteq \overline{x_1)\alpha}^C\ \gamma^{-1}$, where $x_1)\alpha = 0$. Consequently $[\langle 00\rangle\ \ j_2] \in \overline{0}^C\ \gamma^{-1}$, in which case j_2 must be assigned to the $\overline{0}^C$-successor of $\langle 00\rangle$, which from table 4.2(d) is $\langle 01\rangle$. In fact there is a second implication, since figure 4.3(c) shows that state j_1 has state j_3 as $\overline{x_2}^J$-successor, in which case $[\langle 00\rangle\ \ j_3] \in \gamma^{-1}\ \overline{x_2}^J$. Therefore $[\langle 00\rangle\ \ j_3] \in \overline{x_2)\alpha}^C\ \gamma^{-1}$, and $x_2)\alpha = 1$ so $[\langle 00\rangle\ \ j_3] \in \overline{1}^C\ \gamma^{-1}$. Therefore j_3 must be assigned to the $\overline{1}^C$- successor of code $\langle 00\rangle$, and from table 4.2(d) this is $\langle 11\rangle$, so j_3 must be assigned to code $\langle 11\rangle$.

The preceeding shows that the initial assumption, that γ is a one-many weak homomorphism of J to C where $[j_1\ \langle 00\rangle] \in \gamma$, implies $[j_2\ \langle 01\rangle] \in \gamma$ and implies $[j_3\ \langle 11\rangle] \in \gamma$, and this can be expressed as the start of an implication tree, as shown in figure 4.4(a).

$$\begin{tikzcd}
 & {[j_1\ \langle 00\rangle]} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \text{- level one} \\
{[j_2\ \langle 01\rangle]} & & {[j_3\ \langle 11\rangle]} & \text{- level two}
\end{tikzcd}$$

(a) Initial implication tree

$$\begin{tikzcd}
 & & {[j_1\ \langle 00\rangle]} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & & \text{- level one} \\
 & {[j_2\ \langle 01\rangle]} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & & {[j_3\ \langle 11\rangle]} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \text{- level two} \\
\text{—} & {[j_3\ \langle 00\rangle]} & {[j_2\ \langle 00\rangle]} & & {[j_3\ \langle 11\rangle]} & \text{- level three}
\end{tikzcd}$$

(b) Completed implication tree

Figure 4.4

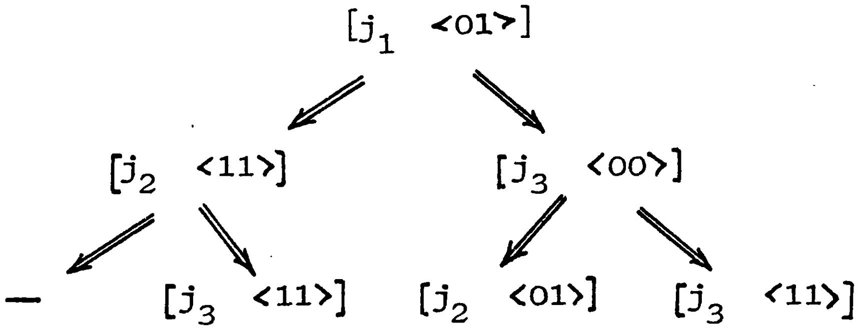
To form the third level of the tree, consider the implied association $[j_2 \quad \langle 01\rangle] \in \gamma$. State symbol j_2 has no $\overline{x_1}^J$ - successor, which means there is no immediate constraint on the $\overline{x_1)\alpha}^C$ - successor of code $\langle 01\rangle$, however the $\overline{x_2}^J$ - successor of j_2 is j_3, so j_3 must be assigned to the $\overline{1}^C$ -successor of $\langle 01\rangle$, which is $\langle 00\rangle$. Thus $[j_2 \quad \langle 01\rangle] \in \gamma$ implies $[j_3 \quad \langle 00\rangle] \in \gamma$, and this implication is shown on the third level of the implication tree of figure 4.4(b). This level can then be completed by considering the association $[j_3 \quad \langle 11\rangle]$ from level two, which implies $[j_2 \quad \langle 00\rangle] \in \gamma$ and $[j_3 \quad \langle 11\rangle] \in \gamma$.

However the tree shows that $[j_1 \quad \langle 00\rangle] \in \gamma$ implies $[j_2 \quad \langle 01\rangle] \in \gamma$, and shows that $[j_2 \quad \langle 01\rangle] \in \gamma$ implies $[j_3 \quad \langle 00\rangle] \in \gamma$, so by transitivity $[j_1 \quad \langle 00\rangle] \in \gamma$ implies $[j_3 \quad \langle 00\rangle] \in \gamma$. Hence the initial assumption, that j_1 is assigned to code $\langle 00\rangle$ by a one-many weak homomorphism γ, implies that j_3 must also be assigned to code $\langle 00\rangle$, in which case γ cannot be one-many. Since the derived association $[j_3 \quad \langle 00\rangle] \in \gamma$ contradicts the initial assumption it must be concluded that the initial assumption is false, that is j_1 cannot be assigned to code $\langle 00\rangle$ by a one-many weak homomorphism under α.

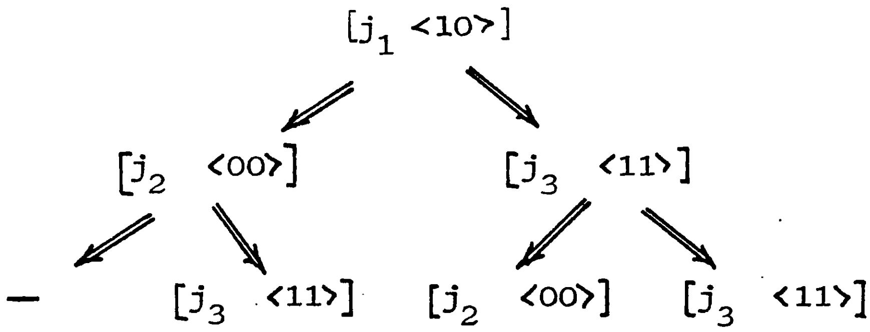
The strategy is that of reductio ad absurdum, and must now be repeated with an alternative initial assumption, such as $[j_1 \quad \langle 01\rangle] \in \gamma$, by tracing the implications. A branch is stopped when an association is encountered for a second time, since continuing the branch will produce implied associations already derived, and if all the branches can be stopped so that each objective state is allocated to some code in S_C, and no contradictions are encountered, the tree will represent a one-many weak homomorphism of J into C under α. If a contradiction arises then the tree is abandoned, a new initial assumption such as $[j_1 \ \langle 10 \rangle] \in \gamma$ must be made and the implications must be traced on a new tree. If this assumption is contradicted, it remains to test the assumption $[j_1 \ \langle 11 \rangle] \in \gamma$. If this assumption fails then the possible assignments $[j_1 \ \langle 00 \rangle] \in \gamma$, $[j_1 \ \langle 01 \rangle] \in \gamma$, $[j_1 \ \langle 10 \rangle] \in \gamma$ and $[j_1 \ \langle 11 \rangle] \in \gamma$ for objective state j_1 have all failed, and a weak homomorphism γ of J to C must have domain $D[\gamma] = S_J$, so it must be concluded that no one-many weak homomorphism of J to C under the input assignment α exists. Then input assignment α can be redefined as $\alpha = \{\langle x_1 \ 1 \rangle \ \langle x_2 \ 0 \rangle\}$, and the process can be repeated. If no successful state assignment can then be found, however, it must be concluded that the stock unit is not a basis for realising the objective automaton.

Thus having derived a contradiction on graph 4.4(b), it is assumed that γ is a one-many weak homomorphism of semiautomaton J to semiautomaton C where $[j_1 \ \langle 01 \rangle] \in \gamma$.

Figure 4.5(a) shows that $[j_1 \ \langle 01\rangle] \in \gamma$ implies $[j_2 \ \langle 11\rangle] \in \gamma$ and $[j_3 \ \langle 11\rangle] \in \gamma$, in which case both j_2 and j_3 are assigned to code $\langle 11\rangle$.



(a) $[j_1 \ \langle 01\rangle] \in \gamma$ implies $[j_2 \ \langle 11\rangle] \in \gamma$ and $[j_3 \ \langle 11\rangle] \in \gamma$



(b) Implication tree without contradiction

Figure 4.5

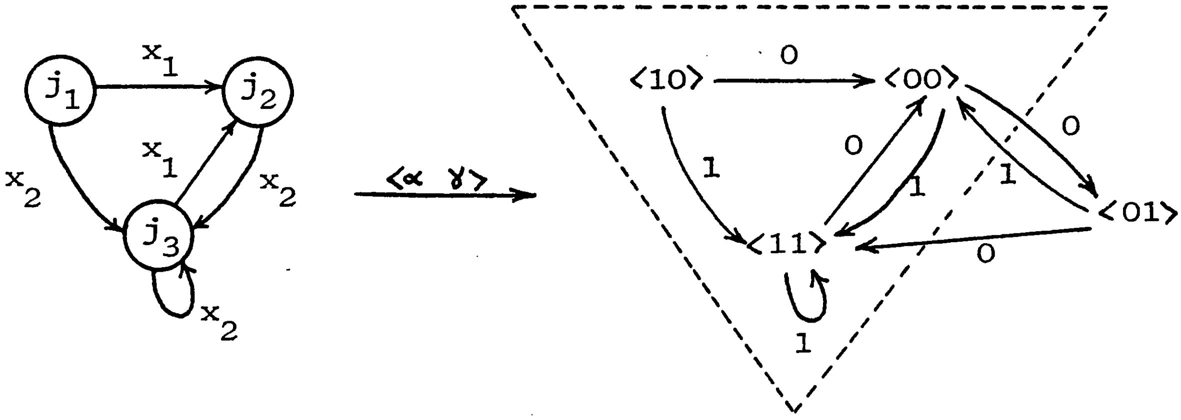
Then the assignment cannot be one-many, so the initial assumption is contradicted and is shown to be invalid. Now it is assumed that γ is a one-many weak homomorphism of J to C where $[j_1 \;\; \langle 10 \rangle] \in \gamma$, and the implications are traced through in figure 4.5(b). Since the associations on level three repeat the entries at preceeding levels, the tree is terminated without a contradiction being derived. Furthermore each objective state has been assigned to some code, so

$D[\gamma] = S_J$, and

$$\gamma = \left\{ [j_1 \;\; \langle 10 \rangle] \;\; [j_2 \;\; \langle 00 \rangle] \;\; [j_3 \;\; \langle 11 \rangle] \right\}$$

is a one-many weak homomorphism of semiautomaton J to semiautomaton C. In fact the assignment γ is one-one, but in general a weak homomorphism determined in this way will be one-many.

If a state assignment γ is a one-many weak homomorphism under an input assignment α, then $\langle \alpha \;\; \gamma \rangle$ will be called an "assignment" of one semiautomaton into the other and the preceeding shows that $\langle \alpha \;\; \gamma \rangle$ is an assignment of objective semiautomaton J into stock semiautomaton **C**. This means that the graph of semiautomaton C can be redrawn to give a graph closely related to that of semiautomaton J, as shown in figure 4.6.



<u>Objective Semiautomaton</u>

<u>$J = \langle S_J\ \overline{X}_J\rangle$</u>

<u>Stock Semiautomaton</u>

<u>$C = \langle S_C\ \overline{X}_C\rangle$</u>

<u>Figure 4.6</u>

The figure shows that the assignment $\langle\alpha\quad\gamma\rangle$ can be used to translate semiautomaton J "into" semiautomaton C, so that the graph obtained by the translation forms part of the graph of the stock semiautomaton. Alternatively the objective states can be considered to be superimposed on the states of stock semiautomaton C, in accordance with the state assignment γ, and every arc on the graph of semiautomaton J will be superimposed on an arc that is labelled in accordance with the input assignment α. It is interesting to observe that code $\langle 01\rangle$ is not involved in the assignment, that is $\langle 01\rangle$ is excluded from the codomain of γ, so this code will not play a part in the realisation of the objective automaton. It is also interesting to observe that j_2 has no $\overline{x_1}^J$-successor. The existence of a $\overline{x_1}^J$ - successor for j_2 would impose a constraint on the $\overline{0}^C$ -successor of code <00> , for example if j_1 was the $\overline{x_1}^J$ - successor of j_2 then <10> would have to be the $\overline{0}^C$ - successor of <00> . The objective automaton would then be complete, and the codes <00> , <10> and <11> would define a subalgebra of stock semiautomaton $C = \langle S_C \ \overline{X}_C \rangle$. In fact a weak homomorphism will always assign a complete objective semiautomaton onto a subalgebra. In the present example, however, the objective state j_2 has no $\overline{x_1}^J$ - successor so the $\overline{0}^C$ - successor of <00> is unrestricted, the $\overline{0}^C$ -successor of <00> could have been <00> , <01> , <10> or <11>, and $\langle \alpha \quad \gamma \rangle$ would still be a valid assignment. Here <01> is the $\overline{0}^C$ -successor of <00> , and the objective semiautomaton is not assigned onto a subalgebra.

The preceeding shows that $\langle \alpha \quad \gamma \rangle$ is an assignment of objective semiautomaton J to stock semiautomaton C, that is γ is a one-many weak homomorphism of J to C under α , in which case the stock unit can be used as a basis for realising the objective automaton. Each output symbol from output set $Z_J = \{z_1, z_2, z_3\}$ must be assigned to a distinct output code, and this will require two output variables z_a, z_b. Then $Z_C = \{$<00> <01> <10> <11>$\}$ represents the possible combinations of the variable values, for example <01> expresses $z_a = 0$ and $z_b = 1$, and each element from the objective output set Z_J can be assigned to a unique code by defining an injection β from Z_J to Z_C, where $D\,[\beta] = Z_J$. Output assignment β can be arbitrary, for example define

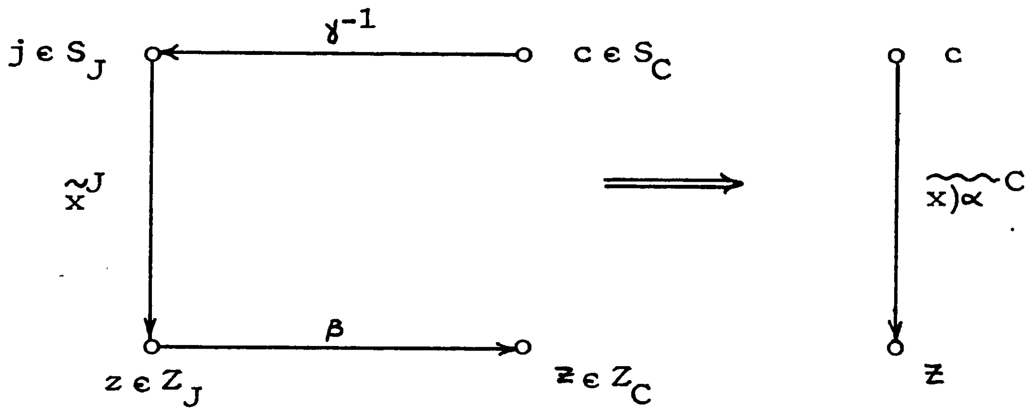
$$\beta = \left\{ [z_1 \ \ \langle 00 \rangle] \ \ [z_2 \ \ \langle 01 \rangle] \ \ [z_3 \ \ \langle 11 \rangle] \right\},$$

but the association of output codes with the arcs of the stock semiautomaton must be carefully considered. The association of output codes with the arcs must be made so that

$$(\forall x)(x \in X_J \implies \gamma^{-1}\, \tilde{x}^J \beta \subseteq \widetilde{x)\alpha}^C),$$

ta

and this commutivity requirement, which has some similarity with the preserved relation concept and with weak homomorphism, is illustrated in figure 4.7.



$\langle c \ \ \bar{z} \rangle \in \gamma^{-1}\, \tilde{x}^J \beta$ implies $\langle c \ \ \bar{z} \rangle \in \widetilde{x)\alpha}^C$

Figure 4.7

The figure shows that objective state j has been assigned to code c by state assignment γ, shows that

$\tilde{x}^J$ associates output symbol z with state j, and shows that output assignment β assigns output symbol z to output code $\mathbf{z}$. Consequently, $\widetilde{x)\alpha}^C$ must be defined so that $\langle c \;\; \mathbf{z} \rangle \in \widetilde{x)\alpha}^C$. These requirements are illustrated in figure 4.8, for example figure 4.8(a) shows
$[\langle 11\rangle \;\; j_3] \in \gamma^{-1}$, $\langle j_3 \;\; z_3\rangle \in \tilde{x}_1^J$ and $[z_3 \;\; \langle 11\rangle] \in \beta$
so $[\langle 11\rangle \;\; \langle 11\rangle] \in \gamma^{-1}\,\tilde{x}_1^J\,\beta$.

$$\langle 00\rangle \xrightarrow{\gamma^{-1}} j_2$$
$$\langle 01\rangle$$
$$\langle 10\rangle \xrightarrow{\gamma^{-1}} j_1$$
$$\langle 11\rangle \xrightarrow{\gamma^{-1}} j_3 \xrightarrow{\tilde{x}_1^J} z_3 \xrightarrow{\beta} \langle 11\rangle \quad \therefore [\langle 11\rangle \;\; \langle 11\rangle] \quad \in \tilde{0}^C$$

(a) $\gamma^{-1}\,\tilde{x}_1^J\,\beta \subseteq \tilde{0}^C$

$$\langle 00\rangle \xrightarrow{\gamma^{-1}} j_2 \xrightarrow{\tilde{x}_2^J} z_1 \xrightarrow{\beta} \langle 00\rangle \quad \therefore [\langle 00\rangle \;\; \langle 00\rangle] \quad \in \tilde{1}^C$$
$$\langle 01\rangle$$
$$\langle 10\rangle \xrightarrow{\gamma^{-1}} j_1$$
$$\langle 11\rangle \xrightarrow{\gamma^{-1}} j_3 \xrightarrow{\tilde{x}_2^J} z_2 \xrightarrow{\beta} \langle 01\rangle \quad \therefore [\langle 11\rangle \;\; \langle 01\rangle] \quad \in \tilde{1}^C$$

(b) $\gamma^{-1}\,\tilde{x}_2^J\,\beta \subseteq \tilde{1}^C$

Figure 4.8

Consequently, $\widetilde{x_1)\alpha}^C = \tilde{0}^C$ must be defined so that $[\langle 11\rangle \;\; \langle 11\rangle] \in \tilde{0}^C$. Similarly figure 4.8(b) shows

$[\langle 00\rangle \quad \langle 00\rangle] \in \gamma^{-1}\,\tilde{x}_2{}^J\beta$, which requires

$[\langle 00\rangle \quad \langle 00\rangle] \in \tilde{1}^C$, and shows

$[\langle 11\rangle \quad \langle 01\rangle] \in \gamma^{-1}\,\tilde{x}_2{}^J\beta$ so $[\langle 11\rangle \quad \langle 01\rangle] \in \tilde{1}^C$.

Then the way the input variable x, and the state variables y_1 and y_2 associated with the stock unit, must produce each of the output variables is shown in figure 4.9.

x \ $\langle y_1y_2\rangle$	$\langle 00\rangle$	$\langle 01\rangle$	$\langle 11\rangle$	$\langle 10\rangle$
0	-	-	1	-
1	0	-	0	-

(a) Karnaugh map showing $z_a = \bar{x}$

x \ $\langle y_1y_2\rangle$	$\langle 00\rangle$	$\langle 01\rangle$	$\langle 11\rangle$	$\langle 10\rangle$
0	-	-	1	-
1	0	-	1	-

(b) Karnaugh map showing $z_b = y_1$

Figure 4.9

For example the requirement $[\langle 11\rangle \quad \langle 11\rangle] \in \tilde{0}^C$ means that state-code $\langle y_1y_2\rangle = \langle 11\rangle$ and input 0 must produce output code $\langle z_az_b\rangle = \langle 11\rangle$, so for

$\langle y_1y_2\rangle = \langle 11\rangle$ and $x = 0$ the z_a-map must have $z_a = 1$, and the z_b-map must have $z_b = 1$. Similarly, the requirement $[\langle 00\rangle \quad \langle 00\rangle] \in \tilde{1}^C$ is satisfied if $z_a = 0$ and $z_b = 0$ for $x = 1$ and $\langle y_1y_2\rangle = \langle 00\rangle$, and $[\langle 11\rangle \quad \langle 01\rangle] \in \tilde{1}^C$ requires $z_a = 0$ and $z_b = 1$ for $x = 1$ and $\langle y_1y_2\rangle = \langle 11\rangle$. Otherwise the output code is unrestricted, and the "don't care" conditions are marked —, giving $z_a = \overline{x}$ and $z_b = y_1$.

The resulting circuit is shown in figure 4.10(a), and consists of the stock unit augmented by combinatorial units deriving the output codes. To confirm the circuit to be a valid realisation of objective automaton $\hat{J}$, the circuit can be formalised as a "circuit" automaton $\hat{C} = \langle S_C\ X_C\ Z_C\ \overline{X}_C\ \tilde{X}_C\rangle$. The associated semiautomaton is the semiautomaton $C = \langle S_C\ \overline{X}_C\rangle$ expressing the state transitions of the stock unit, where $S_C = \{\langle 00\rangle \quad \langle 01\rangle \quad \langle 10\rangle \quad \langle 11\rangle\}$ and $X_C = \{0,1\}$, furthermore $Z_C = \{\langle 00\rangle \quad \langle 01\rangle \quad \langle 10\rangle \quad \langle 11\rangle\}$ is the set of output codes $\langle z_az_b\rangle$, and $\tilde{X}_C = \{\tilde{0}^C, \tilde{1}^C\}$. To define the mapping $\tilde{0}^C : S_C \longrightarrow Z_C$ associated with input 0 consider table (a), which shows the way each state code is converted to an output code when $x = 0$.

$\langle y_1y_2\rangle$	$\langle z_az_b\rangle$
$\langle 00\rangle$	$\langle 10\rangle$
$\langle 01\rangle$	$\langle 10\rangle$
$\langle 10\rangle$	$\langle 11\rangle$
$\langle 11\rangle$	$\langle 11\rangle$

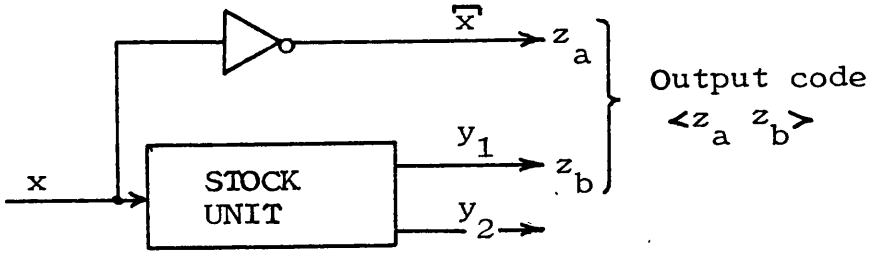
$z_a=\overline{x}=1, z_b=y_1$

(a) Mapping $\tilde{0}^C : S_C \longrightarrow Z_C$

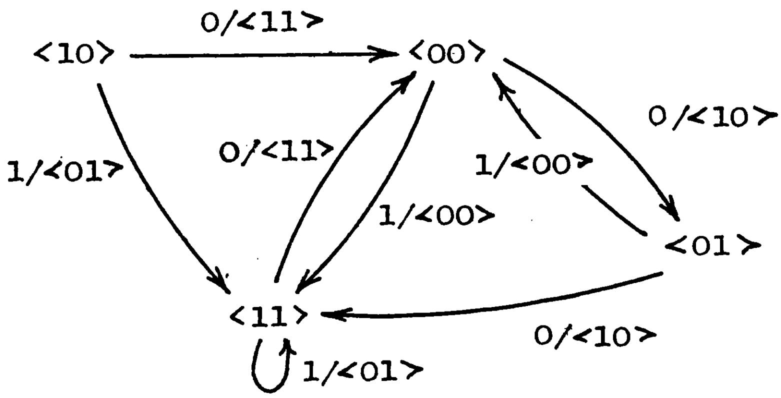
$\langle y_1y_2\rangle$	$\langle z_a\ z_b\rangle$
$\langle 00\rangle$	$\langle 00\rangle$
$\langle 01\rangle$	$\langle 00\rangle$
$\langle 10\rangle$	$\langle 01\rangle$
$\langle 11\rangle$	$\langle 01\rangle$

$z_a=\overline{x}=0, z_b=y_1$

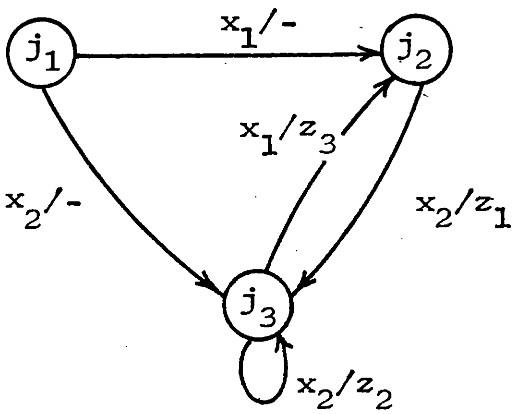
(b) Mapping $\tilde{1}^C : S_C \longrightarrow Z_C$



(a) A realisation of objective automaton $\hat{J}$



(b) Automaton $\hat{C} = \langle S_C \; X_C \; Z_C \; \overline{X}_C \; \tilde{X}_C \rangle$



(c) Objective automaton $\hat{J} = \langle S_J \; X_J \; Z_J \; \overline{X}_J \; \tilde{X}_J \rangle$

Figure 4.10

The $\langle z_a z_b \rangle$ column is formed by observing $z_a = \overline{x}$ and $z_b = y_1$, and the association of an output code with each state code is formalised as the mapping

$\tilde{o}^C : S_C \longrightarrow Z_C$ where

$$\tilde{o}^C = \{[\langle 00\rangle \ \ \langle 10\rangle] [\langle 01\rangle \ \ \langle 10\rangle] [\langle 10\rangle \ \ \langle 11\rangle] [\langle 11\rangle \ \ \langle 11\rangle]\}.$$

Table (b) is formed in the same way, and from the table

$$\tilde{1}^C = \{[\langle 00\rangle \ \ \langle 00\rangle] [\langle 01\rangle \ \ \langle 00\rangle] [\langle 10\rangle \ \ \langle 01\rangle] [\langle 11\rangle \ \ \langle 01\rangle]\} .$$

The automaton $\hat{C} = \langle S_C \ X_C \ Z_C \ \overline{X}_C \ \tilde{X}_C \rangle$ is represented in figure 4.10(b), and it is evident that the automaton is closely related to objective automaton $\hat{J}$. The states of objective automaton $\hat{J}$ on figure (c) can be superimposed on state codes of figure (b) in accordance with state assignment γ , and every arc on the graph of automaton $\hat{J}$ will be superimposed on an arc on the graph of automaton $\hat{C}$. In addition input symbols will be superimposed on input codes in accordance with assignment α , and output symbols will be superimposed on output codes in accordance with β. Furthermore consider any tape from valid event V, for example consider the valid tape $\langle x_1 \ x_2 \ x_1 \ x_2 \rangle$. The graph of automaton $\hat{J}$ shows $\langle j_1 \ j_2 \rangle \in \overline{x}_1^{\,J}$, $\langle j_2 \ j_3 \rangle \in \overline{x}_2^{\,J}$, $\langle j_3 \ j_2 \rangle \in \overline{x}_1^{\,J}$ and $\langle j_2 \ z_1 \rangle \in \tilde{x}_2^{\,J}$, so $\langle j_1 \ z_1 \rangle \in \overline{x}_1^{\,J} \ \overline{x}_2^{\,J} \ \overline{x}_1^{\,J} \ \tilde{x}_2^{\,J}$, that is $\langle j_1 \ z_1 \rangle \in \widetilde{\langle x_1 \ x_2 \ x_1 \ x_2 \rangle}^{J}$. Translating the tape $\langle x_1 \ x_2 \ x_1 \ x_2 \rangle$ under α gives $\langle x_1)\alpha \quad x_2)\alpha \quad x_1)\alpha \quad x_2)\alpha \rangle$, and $x_1)\alpha = 0$, $x_2)\alpha = 1$ so the tape $\langle x_1 \ x_2 \ x_1 \ x_2 \rangle$ from $X_J^{\,*}$ is translated to the tape $\langle 0101 \rangle$ from $X_C^{\,*}$. Furthermore γ assigns objective state j_1 to state code $\langle 10 \rangle$, and the graph of automaton $\hat{C}$ shows

$[\langle 10\rangle \quad \langle 00\rangle] \in \widetilde{\langle 0101\rangle}^{C}$, that is

$[\langle 10\rangle \quad z_1)\beta\ ] \in \widetilde{\langle 0101\rangle}^{C}$. Hence the circuit in state $\langle 10\rangle$ simulates the objective association $\langle j_1\ z_1\rangle \in \widetilde{x_1\ x_2\ x_1\ x_2}^{J}$, by producing output code $z_1)\beta\ = \langle 00\rangle$ in response to the applied input sequence $\langle 0101\rangle$. In fact the circuit in state $\langle 10\rangle$ simulates the objective translation completely, and this can be verified by considering the design approach more formally.

4.3 Realisation Theory

The aim is to present a formal appreciation of automaton realisation, and to begin it is important to establish further symbology. Input assignment α assigns each input symbol x from X_J to a distinct code $x)\alpha$ in X_C, so in the obvious way a tape $t_p = \langle p_1 p_2 \ldots p_u\rangle$ from X_J^* will be converted to a tape $\langle p_1)\alpha\ \ p_2)\alpha\ \ldots p_u)\alpha\rangle$ in X_C^*. More generally a mapping α of a set X_J into a set X_C defines a closely related mapping $\alpha*$ of X_J^* into X_C^*, where $\alpha*$ assigns a tape $t_p = \langle p_1 \ldots p_u\rangle$ from X_J^* to the tape $t_p)\ \alpha* = \langle p_1)\alpha\ \ldots p_u)\alpha\rangle$ in X_C^*, and the mapping $\alpha* : X_J^* \longrightarrow X_C^*$ will then be said to be "generated" by the mapping $\alpha : X_J \longrightarrow X_C$.

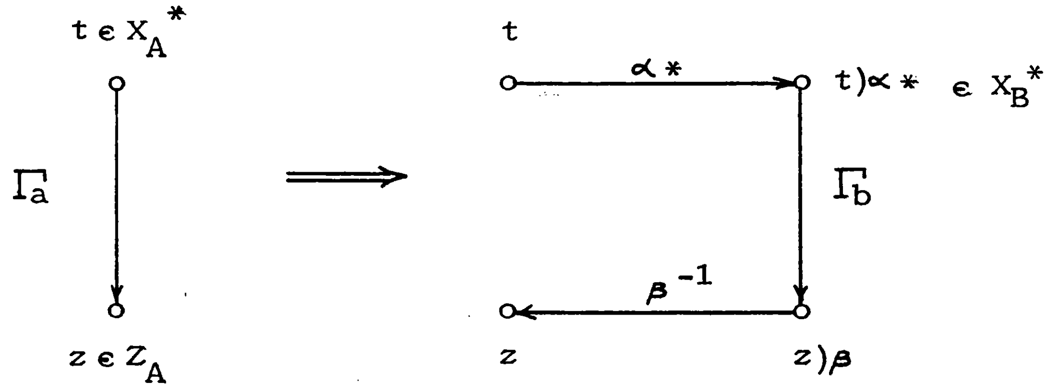
This symbology can now be used to define the important relation of "covering" between automaton states, taking into account an input assignment α and a state assignment β.

<u>Definition</u>

Let $\hat{A} = \langle S_A\ X_A\ Z_A\ \overline{X}_A\ \tilde{X}_A \rangle$, $\hat{B} = \langle S_B\ X_B\ Z_B\ \overline{X}_B\ \tilde{X}_B \rangle$ be automata. A state $a \in S_A$ is <u>covered</u> by a state $b \in S_B$, denoted $a \leqslant b$ or $a \leqslant^{\alpha\beta} b$, iff there exists an injection α of X_A into X_B, and an injection β of Z_A into Z_B, such that for any $t \in X_A^*$ and any $z \in Z_A$,

$$\langle a\ z \rangle \in \tilde{t}^A \implies \langle b\ z)\beta \rangle \in \widetilde{t)\alpha *}^B$$

An alternative definition is readily established since $\langle a\ z \rangle \in \tilde{t}^A$ is equivalent to $\langle t\ z \rangle \in \Gamma_a$, and similarly $\langle b\ z)\beta \rangle \in \widetilde{t)\alpha *}^B$ is equivalent to $\langle t)\alpha *\ \ z)\beta \rangle \in \Gamma_b$. Consequently $a \leqslant^{\alpha\beta} b$ iff $\langle t\ z \rangle \in \Gamma_a$ implies $\langle t)\alpha *\ \ z)\beta \rangle \in \Gamma_b$, or more succinctly $a \leqslant^{\alpha\beta} b$ iff $\Gamma_a \subseteq \alpha * . \Gamma_b . \beta^{-1}$, as shown in figure 4.11.



<u>Figure 4.11</u> <u>$a \leqslant^{\alpha\beta} b$ iff $\Gamma_a \subseteq \alpha * \Gamma_b \beta^{-1}$</u>

In particular let automaton $\hat{B}$ be identical to automaton $\hat{A}$, set $\alpha = \Delta[X_A]$ and set $\beta = \Delta[Z_A]$, so that the state-covering relation becomes a relation over S_A

rather than a relation from one state-set to another. If a_i, a_j are states from S_A where $a_i \leqslant a_j$ then $\Gamma a_i \subseteq \alpha * \Gamma a_j \beta^{-1}$, that is $\Gamma a_i \subseteq \Gamma a_j$. Clearly $a_i \leqslant a_j$, where $a_i, a_j \in S_A$, means that any translation under Γa_i is a translation under Γa_j, so state-mapping Γa_i, and consequently the state a_i, is superfluous. It is also apparent that the state-covering relation is closely related to state-equivalence, since $a_i \equiv a_j$ iff $\Gamma a_i = \Gamma a_j$, consequently $a_i \equiv a_j$ iff $a_i \leqslant a_j$ and $a_j \leqslant a_i$. Furthermore the state-covering relation is closely associated with state-compatibility, since any compatibility class can be replaced by a single covering state. It is particularly important, however, to establish the fundamental nature of the state-covering relation over the state-set S_A, for example it follows from the definition that state-covering is reflexive over S_A, since $a_i \leqslant a_i$ for any state $a_i \in S_A$. Furthermore the relation is transitive over S_A, that is $a_i \leqslant a_j$, $a_j \leqslant a_k$ implies $a_i \leqslant a_k$, for any states $a_i, a_j, a_k \in S_A$. However the relation is not symmetric since $a_i \leqslant a_j$ does not imply $a_j \leqslant a_i$, so the relation is not an equivalence relation but is instead a "quasi-ordering" [Suppes (a)].

It is interesting also to observe that the state-covering relation is "preserved", and this can be verified by considering the distinct automata $\hat{A}$ and $\hat{B}$ of the above definition. Assuming $a \leqslant^{\alpha\beta} b$, let t_p be any tape from X_A^* so that $\langle a\ a' \rangle \in \overline{t_p}^A$ and $\langle b\ b' \rangle \in \overline{t_p)\alpha *}^B$. Then a' is the successor of state $a \in S_A$ for the tape $t_p \in X_A^*$, and b' is the successor of state $b \in S_B$ for the corresponding tape $t_p)\alpha * \in X_B^*$. To confirm $a' \leqslant^{\alpha\beta} b'$, as shown in figure 4.12, assume $t_q \in X_A^*$ where $\langle a'\ z \rangle \in \widetilde{t_q}^A$.

$$
\begin{tikzcd}
a \arrow[r, "\leqslant^{\alpha\beta}"] \arrow[d, "\overline{t_p}^A"'] & b \arrow[d, "\overline{t_p)\alpha *}^B"] \\
a' \arrow[r, phantom, "\Downarrow"] & b'
\end{tikzcd}
$$

$$
\begin{tikzcd}
a' \arrow[r, "\leqslant^{\alpha\beta}"] & b'
\end{tikzcd}
$$

<u>The preserved relation $\leqslant^{\alpha\beta}$</u>

<u>Figure 4.12</u>

Then $\langle a\ a' \rangle \in \overline{t_p}^A$ and $\langle a'\ z \rangle \in \widetilde{t_q}^A$, in which case $\langle a\ z \rangle \in \overline{t_p}^A . \widetilde{t_q}^A$, that is $\langle a\ z \rangle \in \widetilde{t_p \circ t_q}^A$. Furthermore $\langle a\ z \rangle \in \widetilde{t_p \circ t_q}^A$ implies $\langle b\ z)\beta \rangle \in \widetilde{t_p \circ t_q)\alpha *}^B$, since $a \leqslant^{\alpha\beta} b$, and

$$
\begin{aligned}
\widetilde{t_p \circ t_q)\alpha *}^B &= \widetilde{t_p)\alpha * \circ t_q)\alpha *}^B \\
&= \overline{t_p)\alpha *}^B . \widetilde{t_q)\alpha *}^B
\end{aligned}
$$

so $\langle b\ z)\beta\rangle \in \overline{t_p)\alpha *}^{B} . \widetilde{t_q)\alpha *}^{B}$. Therefore $\langle b\ b''\rangle \in \overline{t_p)\alpha *}^{B}$ and $\langle b''\ z)\beta\rangle \in \widetilde{t_q)\alpha *}^{B}$ for some b'', however $\overline{t_p)\alpha *}^{B}$ is a mapping so $\langle b\ b'\rangle \in \overline{t_p)\alpha *}^{B}$, $\langle b\ b''\rangle \in \overline{t_p)\alpha *}^{B}$ implies $b' = b''$. Consequently, $\langle b''\ z)\beta\rangle \in \widetilde{t_q)\alpha *}^{B}$ becomes $\langle b'\ z)\beta\rangle \in \widetilde{t_q)\alpha *}^{B}$. Therefore $\langle a'\ z\rangle \in \tilde{t}_q^{\,A}$ implies $\langle b'\ z)\beta\rangle \in \widetilde{t_q)\alpha *}^{B}$, and $t_q \in X_A^{*}$ is arbitrary so $a' \preccurlyeq^{\alpha\beta} b'$. This shows that state-covering is preserved for any given tape from X_A^{*}, and $X_A \subseteq X_A^{*}$ so state-covering is preserved for any given input $x \in X_A$. Consequently the state-covering relation is preserved "from" semiautomaton $A = \langle S_A\ \overline{X_A}\rangle$ to semiautomaton $B = \langle S_B\ \overline{X_B}\rangle$, and this is an important extension of the preserved relation concept. The state-covering relation has introduced the idea of "extrinsic" preservation, or preservation from one semiautomaton to another, whereas state-equivalence and state-compatibility were regarded as preserved relations "within" a given semiautomaton.

Reconsidering the objective automaton $\hat{J}$, and the automaton $\hat{C}$ representing the realisation, the state-covering relation can be used to express the way the objective state j_1 is related to the state $\langle 10\rangle$ of the realisation. In fact $j_1 \preccurlyeq^{\alpha\beta} \langle 10\rangle$, that is $\Gamma_{j_1} \subseteq \alpha * \Gamma_{\langle 10\rangle}\beta^{-1}$, and this formalises the observation that the objective translation $\Gamma_{j_1} : X_J^{*} \longrightarrow Z_J$ is "simulated" by the circuit in state $\langle 10\rangle$. The design procedure provides at least one covering state for each objective state, so the relation of state-covering from S_J to S_C has domain $D[\leqslant^{\alpha\beta}] = S_J$. Then a mapping χ of S_J into S_C can be derived from the relation $\leqslant^{\alpha\beta}$, so that $\langle j\ c \rangle \in \chi$ implies $j \leqslant^{\alpha\beta} c$, in which case automaton $\hat{J}$ is said to be "covered" by automaton $\hat{C}$.

Definition

An automaton $\hat{A} = \langle S_A\ X_A\ Z_A\ \overline{X}_A\ \tilde{X}_A \rangle$ is covered by an automaton $\hat{B} = \langle S_B\ X_B\ Z_B\ \overline{X}_B\ \tilde{X}_B \rangle$, denoted $\hat{A} \leqslant \hat{B}$ or $\hat{A} \leqslant^{\alpha\beta\chi} \hat{B}$, iff there is an injection α of X_A into X_B, an injection β of Z_A into Z_B and a mapping χ of S_A into S_B, such that $\langle a\ b \rangle \in \chi \implies a \leqslant^{\alpha\beta} b$

The present aim is to show that the design procedure, as used in formulating the realisation $\hat{C}$ of objective automaton $\hat{J}$, ensures that $\hat{J}$ is covered by $\hat{C}$. Since the approach was based on a weak homomorphism of semiautomaton $J = \langle S_J\ \overline{X}_J \rangle$ to semiautomaton $C = \langle S_C\ \overline{X}_C \rangle$, let $J \xrightarrow{\alpha\gamma} C$ denote that γ is a weak homomorphism of J into C under an input assignment α. That is, $J \xrightarrow{\alpha\gamma} C$ denotes that α is an injection of X_J into X_C, and γ is a relation from S_J to S_C with domain $D[\gamma] = S_J$, so that

$$(\forall x)(x \in X_J \implies \gamma^{-1}\ \bar{x}^J \subseteq \overline{x)\alpha}^C\ \gamma^{-1})$$

In particular, the design approach was based on a "one-many" weak homomorphism of semiautomaton J into semiautomaton C, in which case J is said to be "covered" by C.

Definition

A semiautomaton $A = \langle S_A \ \overline{X}_A \rangle$ is covered by a semiautomaton $B = \langle S_B \ \overline{X}_B \rangle$, denoted $A \leqslant B$ or $A \leqslant^{\alpha\gamma} B$, iff $A \xrightarrow{\alpha\gamma} B$ where γ is one-many.

The problem of realising an objective automaton $\hat{J}$ reduces to that of formulating a semiautomaton C representing the state-transitions of a circuit, so that $J \leqslant C$. The semiautomaton C can then be used as the basis of an automaton $\hat{C}$ where $\hat{J} \leqslant \hat{C}$, since this approach ensures that each output mapping $\widetilde{x)\alpha}^{C}$ can be formalised so that $\gamma^{-1} \tilde{x}^{J} \beta \subseteq \widetilde{x)\alpha}^{C}$, as in figure 4.7.

Theorem [cf. Ginzburg; Yoeli]

Let $\hat{J}$ be an automaton with semiautomaton J, and let C be a semiautomaton where $J \leqslant C$. Then there exists an automaton $\hat{C}$ with semiautomaton C, such that $\hat{J} \leqslant \hat{C}$.

Proof

Since $J \leqslant C$ assume $J \leqslant^{\alpha\gamma} C$, that is assume $J \xrightarrow{\alpha\gamma} C$ where γ is one-many. Then α is an injection of X_J into X_C, and γ is a relation from S_J to S_C where

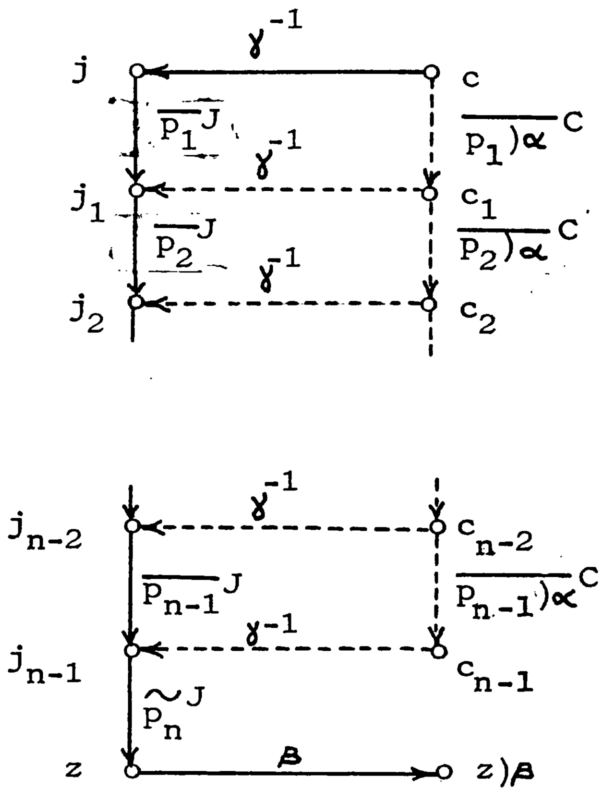
$D[\gamma] = S_J$ and $(\forall x)(x \in X_J \implies \gamma^{-1} \bar{x}^{J} \subseteq \overline{x)\alpha}^{C} \gamma^{-1})$

Let β be an arbitrary injection of Z_J into a finite set $Z_C = \{z_1, z_2, z_3, \ldots\}$. If α assigns input symbol $x \in X_J$ to symbol $x)\alpha \in X_C$ then associate with $x)\alpha$ a mapping $\widetilde{x)\alpha}^{C}$ from S_C to Z_C, such that $\gamma^{-1} \tilde{x}^{J} \beta \subseteq \widetilde{x)\alpha}^{C}$. Since γ is one-many γ^{-1} is a mapping, and this ensures that an appropriate mapping $\widetilde{x)\alpha}^{C}$ can be defined, indeed set $\gamma^{-1}\widetilde{x}^J\beta = \widetilde{x)\alpha}^C$ if desired. Let $\widetilde{X}_C$ denote the set of all the mappings $\widetilde{x)\alpha}^C$ where $x \in X_J$, and define $\hat{C} = \langle S_C\ X_C\ Z_C\ \bar{X}_C\ \widetilde{X}_C \rangle$.

Let χ be an arbitrary injection derived from relation γ, that is assume χ is an injection where $\chi \subseteq \gamma$ and $D[\chi] = D[\gamma]$, in which case $D[\chi] = S_J$ since $D[\gamma] = S_J$. Furthermore let σ denote the mapping of X_J^* into X_C^* generated by mapping $\alpha : X_J \longrightarrow X_C$, that is assume $\sigma = \alpha*$.

Assume $\langle j\ c \rangle \in \chi$, and in order to show that this implies $j \leqslant c$ assume $t_p \in X_J^*$ where $t_p = \langle p_1 p_2 \cdots p_{u-1}\, p_u \rangle$, and assume $\langle j\ z \rangle \in \widetilde{t_p}^J$. Then $\langle j\ z \rangle \in \bar{p}_1^J\ \bar{p}_2^J \cdots \overline{p_{u-1}}^J\ \widetilde{p}_u^J$, in which case there must exist a sequence $\langle j\ j_1 \rangle \in \bar{p}_1^J$, $\langle j_1\ j_2 \rangle \in \bar{p}_2^J$, $\langle j_2\ j_3 \rangle \in \bar{p}_3^J, \ldots$. $\langle j_{u-2}\ j_{u-1} \rangle \in \overline{p_{u-1}}^J$, $\langle j_{u-1}\ z \rangle \in \widetilde{p}_u^J$. Furthermore $\langle j\ c \rangle \in \chi$ implies $\langle j\ c \rangle \in \gamma$, since $\chi \subseteq \gamma$, and then $\langle c\ j \rangle \in \gamma^{-1}$ so $\langle c\ j_1 \rangle \in \gamma^{-1}\ \bar{p}_1^J$. Since γ is a weak homomorphism $\gamma^{-1}\ \bar{p}_1^J \subseteq \overline{p_1)\alpha}^C\ \gamma^{-1}$, consequently $\langle c\ j_1 \rangle \in \overline{p_1)\alpha}^C\ \gamma^{-1}$, in which case $\langle c\ c_1 \rangle \in \overline{p_1)\alpha}^C$ and $\langle c_1\ j_1 \rangle \in \gamma^{-1}$ for some $c_1 \in S_C$. Then $\langle c_1\ j_1 \rangle \in \gamma^{-1}$ where $\langle j_1\ j_2 \rangle \in \bar{p}_2^J$, giving $\langle c_1\ j_2 \rangle \in \gamma^{-1}\ \bar{p}_2^J$, and $\gamma^{-1}\ \bar{p}_2^J \subseteq \overline{p_2)\alpha}^C\ \gamma^{-1}$ so $\langle c_1\ j_2 \rangle \in \overline{p_2)\alpha}^C\ \gamma^{-1}$, in which case $\langle c_1\ c_2 \rangle \in \overline{p_2)\alpha}^C$ and $\langle c_2\ j_2 \rangle \in \gamma^{-1}$ for some c_2. By continuing this reasoning there must be a sequence

$\langle c\ c_1\rangle \in \overline{p_1)\alpha}^{\,C}$, $\langle c_1\ c_2\rangle \in \overline{p_2)\alpha}^{\,C}$,
$\langle c_2\ c_3\rangle \in \overline{p_3)\alpha}^{\,C}, \ldots .\ \langle c_{n-2}\ c_{n-1}\rangle \in \overline{p_{n-1})\alpha}^{\,C}$ where
$\langle c\ j\rangle,\ \langle c_1\ j_1\rangle,\ \langle c_2\ j_2\rangle, \ldots .\ \langle c_{n-1}\ j_{n-1}\rangle \in \gamma^{-1}$.



Furthermore $D[\beta] = Z_J$, so $\langle z\ z)\beta\rangle \in \beta$ for some $z)\beta \in Z_C$, and from above $\langle c_{n-1}\ j_{n-1}\rangle \in \gamma^{-1}$ and $\langle j_{n-1}\ z\rangle \in \widetilde{p_n}^{\,J}$ so $\langle c_{n-1}\ z)\beta\rangle \in \gamma^{-1}\,\widetilde{p_n}^{\,J}\beta$. In addition $\widetilde{p_n)\alpha}^{\,C}$ has been defined to satisy
$\gamma^{-1}\,\widetilde{p_n}^{\,J}\beta \subseteq \widetilde{p_n)\alpha}^{\,C}$, so $\langle c_{n-1}\ z)\beta\rangle \in \widetilde{p_n)\alpha}^{\,C}$.

Therefore $\langle c\ c_1\rangle \in \overline{p_1)\alpha}^{\,C}$, $\langle c_1\ c_2\rangle \in \overline{p_2)\alpha}^{\,C}, \ldots .$ $\langle c_{n-2}\ c_{n-1}\rangle \in \overline{p_{n-1})\alpha}^{\,C}$ and $\langle c_{n-1}\ z)\beta\rangle \in \widetilde{p_n)\alpha}^{\,C}$, in which case $\langle c\ z)\beta\rangle \in \overline{p_1)\alpha}^{\,C}\ \overline{p_2)\alpha}^{\,C} \ldots .\overline{p_{n-1})\alpha}^{\,C}\ \widetilde{p_n)\alpha}^{\,C}$, and $\langle p_1)\alpha\ \ p_2)\alpha\ \ldots .p_{n-1})\alpha\ \ p_n)\alpha\rangle = t_p)\sigma$ so
$\widetilde{t_p)\sigma}^{\,C} = \overline{p_1)\alpha}^{\,C}\ \overline{p_2)\alpha}^{\,C} \ldots .\overline{p_{n-1})\alpha}^{\,C}\ \widetilde{p_n)\alpha}^{\,C}$, therefore
$\langle c\ z)\beta\rangle \in \widetilde{t_p)\sigma}^{\,C}$.

Hence $\langle j\ z\rangle \in \widetilde{t_p}^J$ implies $\langle c\ z)\beta\rangle \in \widetilde{t_p)\sigma}^C$, and $t_p \in X_J^*$ is arbitrary so $j \leqslant^{\alpha\beta} c$. Therefore $\langle j\ c\rangle \in \chi$ implies $j \leqslant^{\alpha\beta} c$, confirming $\hat{J} \leqslant^{\alpha\beta\chi} \hat{C}$ and completing the proof.

The theorem can be applied, in particular, in realising a given objective automaton using stock units. The approach is to represent a given stock unit as a semiautomaton, and to seek an input assignment α and a one-many state-assignment γ such that γ is a weak homomorphism under α. If such an assignment $\langle\alpha\ \gamma\rangle$ can be found, the theorem ensures that the stock unit can be supplemented by combinatorial units to generate output codes. For example the transition tree of figure 4.5(b) reveals an assignment $\langle\alpha\ \gamma\rangle$ of objective semiautomaton J to the semiautomaton C, and this ensures that the stock unit represented as C can be used as a basis for realising the objective automaton. In this example the weak homomorphism

$$\gamma = \{[j_1\ \ \langle 10\rangle]\ [j_2\ \ \langle 00\rangle]\ [j_3\ \ \langle 11\rangle]\}$$

is injective, so only one injection $\chi = \gamma$ can be derived. Then from the theorem $\langle j\ c\rangle \in \chi$ implies $j \leqslant c$, and in particular $j_1 \leqslant \langle 10\rangle$, $j_2 \leqslant \langle 00\rangle$, $j_3 \leqslant \langle 11\rangle$.

4.4 Automaton Reduction

Various authors have considered the problem of reducing a partial automaton to fewer states [Ginsburg; Grasselli & Luccio; Kella; Paull & Unger], and the conventional approach to automaton realisation should always be preceeded by a reduction study. This is not the case, however, when stock units are considered as realisation components, and realisation should then be attempted without reduction. This can be appreciated once the "reduction" concept is formalised, the idea being that a "reduction" covers the original automaton but has fewer states.

<u>Definition</u> [Yoeli]

An automaton $\hat{B} = \langle S_B\ X_B\ Z_B\ \overline{X}_B\ \tilde{X}_B \rangle$ is a <u>reduction</u> of an automaton $\hat{A} = \langle S_A\ X_A\ Z_A\ \overline{X}_A\ \tilde{X}_A \rangle$ iff $\hat{A} \leqslant \hat{B}$ and S_B is a cover of S_A.

A reduction of a given automaton $\hat{A}$ can be found by forming the "final class" π_A of the automaton, because the final class is the maximal "output-consistent" preserved cover.

<u>Definition</u>

Let $\hat{A} = \langle S_A\ X_A\ Z_A\ \overline{X}_A\ \tilde{X}_A \rangle$ be an automaton and let π be a cover of S_A. The cover π is <u>output-consistent</u> iff

$$(\forall x)(\forall B)(x \in X_A, B \in \pi \Longrightarrow (B)\tilde{x}^A = \emptyset \text{ or } (B)\tilde{x}^A \text{ is a singleton})$$

The final class π_A of the automaton $\hat{A}$ is the cover defined by the relation of state-compatibility over S_A, so each block of cover π_A consists of mutually compatible states. To confirm π_A to be output-consistent assume $x \in X_A$ and

$B \in \pi_A$ where $(B)\tilde{x}^A \neq \emptyset$, so $\langle b\ z_i \rangle \in \tilde{x}^A$ for some $b \in B$ and some $z_i \in Z_A$. Let b' denote any element of B, and assume $\langle b'\ z_j \rangle \in \tilde{x}^A$. Then $\langle b\ z_i \rangle \in \tilde{x}^A$ and $\langle b'\ z_j \rangle \in \tilde{x}^A$, in which case $z_i = z_j$ since b and b' are compatible states, consequently $(B)\tilde{x}^A = \{z_i\}$. Hence either $(B)\tilde{x}^A = \emptyset$ or $(B)\tilde{x}^A$ is a singleton, so π_A is output-consistent.

Furthermore the final class is preserved, since state-compatibility is a preserved relation, consequently the final class can be used to form a reduction.

Result [Yoeli]

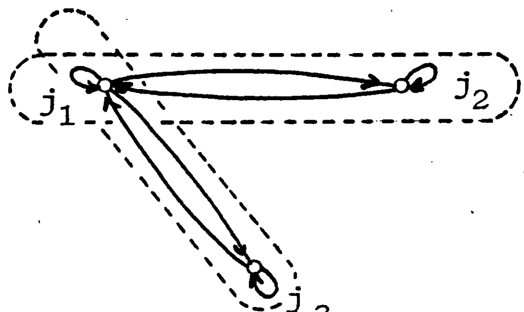
Let $\hat{A} = \langle S_A\ X_A\ Z_A\ \overline{X}_A\ \tilde{X}_A \rangle$ be an automaton, and let π be a cover of S_A where π is output-consistent and is preserved within $A = \langle S_A\ \overline{X}_A \rangle$. Let $G = \langle S_G\ \overline{X}_G \rangle$ be an arbitrary π-image of semiautomaton A.

Then there exists a reduction $\hat{G} = \langle S_G\ X_G\ Z_G\ \overline{X}_G\ \tilde{X}_G \rangle$ of automaton A.

This approach will now be used to seek a reduction of the objective automaton $\hat{J}$ from previously, and for convenience the details are repeated as figure 4.13(a).

	x_1	x_2
j_1	$j_2/-$	$j_3/-$
j_2	$-/-$	j_3/z_1
j_3	j_2/z_3	j_3/z_2

(a) Objective Automaton $\hat{J} = \langle S_J\ X_J\ Z_J\ \overline{X}_J\ \tilde{X}_J \rangle$



(b) The state-compatibility relation over S_J

Figure 4.13

The first problem is to find the final class $\Pi_{\mathcal{J}}$ associated with automaton $\hat{J}$, and various procedures have been suggested [Das & Sheng; Marcus; Paull & Unger; Sinha Roy & Sheng]. In particular, the final class can be formed by recognising the maximal complete polygons on the graph of the state-compatibility relation over S_J [Kohavi]. However the final class is the maximal output-consistent preserved cover, and this provides an alternative approach. Clearly $(j_1\ j_2\ j_3)$ represents the maximal cover of S_J, however this S_J-cover is not output-consistent since $\langle j_2\ z_1 \rangle \ \epsilon\ \tilde{x}_2^J$ whereas $\langle j_3\ z_2 \rangle \ \epsilon\ \tilde{x}_2^J$, giving $\{z_1, z_2\}$ as the $\tilde{x}_2^J$-image of block $(j_1\ j_2\ j_3)$. Putting j_2 and j_3 into separate blocks will form the S_J-cover $(j_1\ j_2)(j_1\ j_3)$, and this is the maximal output-consistent S_J-cover. Since the maximal output-consistent cover is not necessarily preserved, further refinement might be necessary to form the maximal output-consistent preserved cover. However $(j_1\ j_2)(j_1\ j_3)$ is preserved within objective semiautomaton $J = \langle S_J\ \overline{X}_J \rangle$, so this is the maximal output-consistent preserved cover, and the final class of automaton $\hat{J}$ is therefore $\Pi_{\mathcal{J}} = (j_1\ j_2)(j_1\ j_3)$. For interest, the graph of the state-compatibility relation is shown in figure 4.13(b). Clearly $j_1 \approx j_2$ and $j_1 \approx j_3$, whereas j_2 and j_3 are incompatible.

A reduction can be based on the automaton final class, since this is an output-consistent preserved cover, however an output-consistent preserved cover with fewer blocks might exist and this will give a reduction with fewer states. Consequently, attention is directed to refinements of the final class. Since any such refinement will be output-consistent, the problem is to find a refinement of the automaton final class which has fewer blocks and is also preserved. In fact final class $\pi_{\mathcal{J}} = (j_1\ j_2)(j_1\ j_3)$ is the only nontrivial output-consistent preserved cover of S_J, and is the only basis for reducing automaton $\hat{J}$.

Furthermore just one $\pi_{\mathcal{J}}$-image $J/\pi_{\mathcal{J}} = \langle S_{\mathcal{J}}\ \overline{X}_{\mathcal{J}} \rangle$ of semiautomaton $J = \langle S_J\ \overline{X}_J \rangle$ can be formed, where $S_{\mathcal{J}} = \pi_{\mathcal{J}}$, $\overline{X}_{\mathcal{J}} = \left\{ \overline{x}_1^{\mathcal{J}}, \overline{x}_2^{\mathcal{J}} \right\}$, and the mappings $\overline{x}_1^{\mathcal{J}}$, $\overline{x}_2^{\mathcal{J}}$ are those of figure 4.14, with B_1 representing block $(j_1\ j_2)$ of cover $\pi_{\mathcal{J}}$ and B_2 representing block $(j_1\ j_3)$.

(a) Mapping $\overline{x}_1^{\mathcal{J}}$ over $\pi_{\mathcal{J}}$ (b) Mapping $\overline{x}_2^{\mathcal{J}}$ over $\pi_{\mathcal{J}}$

Figure 4.14

In general a given preserved cover will define several image semiautomata, and any of the images associated with an output-consistent preserved cover can be used in forming a reduction.

It remains to use the image $J/\pi_{\mathcal{J}}$ to formalise a reduction $\hat{\mathcal{J}}$ of automaton $\hat{J}$, the procedure being based on the preceeding result. Define $\hat{\mathcal{J}} = \langle S_{\mathcal{J}}\ X_{\mathcal{J}}\ Z_{\mathcal{J}}\ \overline{X}_{\mathcal{J}}\ \widetilde{X}_{\mathcal{J}} \rangle$ where $Z_{\mathcal{J}} = Z_J = \left\{ z_1, z_2, z_3 \right\}$, and define the set $\widetilde{X}_{\mathcal{J}} = \left\{ \widetilde{x}_1^{\mathcal{J}}, \widetilde{x}_2^{\mathcal{J}} \right\}$ of mappings over $S_{\mathcal{J}} = \pi_{\mathcal{J}}$ so that for any

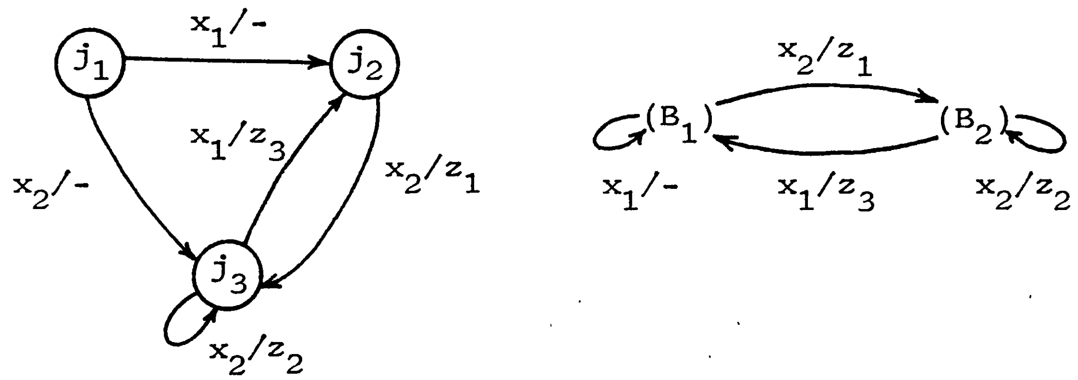
$B \in S_{\mathcal{J}}$ and any $z \in Z_{\mathcal{J}}$,

$$\langle B\ z\rangle \in \tilde{x}_1^{\mathcal{J}} \quad \text{iff} \quad (B)\,\tilde{x}_1^{J} = \{z\}$$

and $\langle B\ z\rangle \in \tilde{x}_2^{\mathcal{J}}$ iff $(B)\,\tilde{x}_2^{J} = \{z\}$.

For example $(B_1)\,\tilde{x}_1^{J} = \emptyset$ and $(B_2)\,\tilde{x}_1^{J} = \{z_3\}$ so $\tilde{x}_1^{\mathcal{J}} = \{\langle B_2\ z_3\rangle\}$, similarly $(B_1)\,\tilde{x}_2^{J} = \{z_1\}$ and $(B_2)\,\tilde{x}_2^{J} = \{z_2\}$ so $\tilde{x}_2^{\mathcal{J}} = \{\langle B_1\ z_1\rangle\ \ \langle B_2\ z_2\rangle\}$. Clearly the definitions for the mappings $\tilde{x}_1^{\mathcal{J}}$, $\tilde{x}_2^{\mathcal{J}}$ require the cover of S_J to be output-consistent.

This completes the formalisation of reduction $\hat{\mathcal{J}} = \langle S_{\mathcal{J}}\ X_{\mathcal{J}}\ Z_{\mathcal{J}}\ \overline{X}_{\mathcal{J}}\ \tilde{X}_{\mathcal{J}}\rangle$, and in figure 4.15 reduction $\hat{\mathcal{J}}$ is compared with objective automaton $\hat{J}$. Each "state" of reduction $\hat{\mathcal{J}}$ covers the related objective states, for example $B_1 = \{j_1\ j_2\}$ so $j_1 \leqslant B_1$ and $j_2 \leqslant B_1$.



(a) Objective Automaton $\hat{J} = \langle S_J\ X_J\ Z_J\ \overline{X}_J\ \tilde{X}_J\rangle$

(b) Reduction $\hat{\mathcal{J}} = \langle S_{\mathcal{J}}\ X_{\mathcal{J}}\ Z_{\mathcal{J}}\ \overline{X}_{\mathcal{J}}\ \tilde{X}_{\mathcal{J}}\rangle$

Figure 4.15

Then $\Gamma_{j_1} \subseteq \Gamma_{B_1}$, for example $[\langle x_1x_2x_1\rangle \;\; z_3] \in \Gamma_{j_1}$ on the graph of automaton $\hat{J}$ so $[\langle x_1x_2x_1\rangle \;\; z_3] \in \Gamma_{B_1}$ on the graph of the reduction $\hat{\mathcal{J}}$. However, j_1 and B_1 are not equivalent states, that is Γ_{j_1} is not identical to Γ_{B_1}, for example $[\langle x_2\rangle \;\; z_1] \in \Gamma_{B_1}$ whereas $\langle x_2\rangle \notin D[\Gamma_{j_1}]$. Since $\Gamma_{j_1} \subseteq \Gamma_{B_1}$ the reduction $\hat{\mathcal{J}}$ does not express the objective translation $\Gamma_{j_1} : X_J^* \longrightarrow Z_J$, but expresses the "superior" translation Γ_{B_1}.

Reduction $\hat{\mathcal{J}}$ can be used to give a realisation of automaton $\hat{J}$, and the approach is to find a stock semiautomaton C so that semiautomaton $\mathcal{J}$ is assigned to C by a one-many weak homomorphism. The semiautomaton C can then be used to form an automaton $\hat{C}$ where $\hat{\mathcal{J}} \leqslant \hat{C}$, automaton $\hat{C}$ being based on semiautomaton C in the normal way. Then $\hat{J} \leqslant \hat{\mathcal{J}}$ and $\hat{\mathcal{J}} \leqslant \hat{C}$, and covering is a transitive relation between automata so $\hat{J} \leqslant \hat{C}$. The objective automaton has been realised "indirectly", by using a one-many weak homomorphism to form a realisation of the reduction $\hat{\mathcal{J}}$. Furthermore, a stock unit might be useful in realising an objective automaton in this indirect way, without being a useful component in forming a direct realisation. There might be a one-many weak homomorphism of semiautomaton $\mathcal{J}$ to a stock semiautomaton C, giving an indirect realisation of the objective automaton, but no one-many weak homomorphism of the objective semiautomaton to the stock semiautomaton.

This does not mean, however, that realisation possibilities will always be increased by reduction, since realisation possibilities can also be lost. An example is provided by the objective automaton $\hat{J}$, and the reduction $\hat{\mathcal{J}}$, as shown in figure 4.15. It has been seen that a one-many weak homomorphism relates the objective semiautomaton J to the stock semiautomaton C of figure 4.2, and figure 4.10 has shown that semiautomaton C can be used to form a realisation $\hat{C}$. However it is easily confirmed, by forming implication trees, that semiautomaton $\mathcal{J}$ is not related to the stock semiautomaton C by a one-many weak homomorphism, so the stock unit cannot be used in realising the reduction. In effect, reduction has lost the possibility of using this stock unit. The general case is shown in figure 4.16, where an objective semiautomaton J is related to a stock semiautomaton C by a one-many weak homomorphism γ, so that the stock unit can be used in realising the objective automaton $\hat{J}$.

$$\begin{tikzcd}
J \arrow[r, "\gamma", "\text{(one-many)}"'] \arrow[d, "\text{(natural)}"] & C \\
\mathcal{J} &
\end{tikzcd}$$

<u>Figure 4.16</u>

If a reduction $\hat{\mathcal{J}}$ is based on an image-semiautomaton $\mathcal{J}$, the objective semiautomaton J will be related to $\mathcal{J}$ by a natural weak homomorphism. However $\mathcal{J}$ might not be related to C by a one-many weak homomorphism, and it may be that the stock unit cannot be used in realising the reduction $\hat{\mathcal{J}}$.

Clearly realisation prospects are changed when an automaton is reduced, and it is concluded that the realisation prospects of an objective automaton, and the realisation prospects of the various reductions, will usually be unrelated. In contrast let $\hat{J} = \langle S_J\ X_J\ Z_J\ \bar{X}_J\ \tilde{X}_J \rangle$ be an arbitrary objective automaton where the objective semiautomaton is complete, and assume that state-equivalence, rather than state-compatibility, is used in forming a reduction. The reduction must then be based on the image semiautomaton J/π_E, where π_E is the equivalence partition associated with automaton $\hat{J}$, and the canonical relation π'_E from S_J to the partition π_E will be a mapping and a weak homomorphism, indeed $J \xrightarrow{\pi'} J/\pi_E$ (epimorphism) since J is complete. Certainly π'_E is a weak homomorphism of J to J/π_E, as shown in figure 4.17, and then ${\pi'_E}^{-1}$ must be a weak homomorphism of J/π_E to J, since the inverse of a weak homomorphism of a complete semiautomaton onto one of common index (here $X_J = X_{\mathcal{J}}$) is itself a weak homomorphism [cf. Yoeli].

$$\begin{tikzcd}
J \arrow[d, "\pi'_E"] \arrow[dr, "\langle \alpha\ \gamma \rangle"] & \\
J/\pi_E \arrow[u, bend left, dashed, "{\pi'_E}^{-1}"] \arrow[r, "\langle \alpha\ \delta \rangle"'] & C
\end{tikzcd}$$

<u>Figure 4.17</u>

Assume, as in figure 4.17, that an assignment $\langle \alpha \quad \gamma \rangle$ relates the objective semiautomaton J to a stock semiautomaton C, so that the stock unit can be used in realising the objective automaton. Then $\pi'^{-1}_E \gamma$ must be a weak homomorphism under the input assignment α, furthermore π'_E is a mapping so π'^{-1}_E must be one-many, in which case $\pi'^{-1}_E \gamma$ must be one-many. Consequently $\langle \alpha , \pi'^{-1}_E \gamma \rangle$ must be an assignment of J/π_E to C, and the stock unit can be used in realising the reduction. Conversely if $\langle \alpha \quad \delta \rangle$ is an assignment of J/π'_E to a stock semiautomaton C, as shown in figure 4.17, then $\pi'_E \delta$ must be a weak homomorphism of J to C under the input assignment α. However $\pi'_E \delta$ will not usually be one-many, since π'_E will be a mapping, and $\langle \alpha , \pi'_E \delta \rangle$ will not usually be an assignment of J to C, so the stock unit might not be useful in forming a direct realisation of the objective automaton.

This shows, in the case of a transition-complete automaton, that realisation prospects might be increased by reduction, furthermore figure 4.17 shows that realisation prospects will not be lost. Consequently a transition-complete objective automaton can be reduced with confidence, so long as state-equivalence is used instead of state-compatibility.

4.5. Conclusion

The conventional approach to automaton realisation is given in various texts [Lewin; Miller], however the designer does not always seek a realisation in the conventional form, as an interconnection of bistables and combinatorial units. For example the designer might seek a linear realisation [Davis (a)], in particular the designer might seek a "degenerate" linear realisation, where the state variables do not require an associated combinatorial system [Davis (b)]. Alternatively, the designer might seek a realisation using a read-only memory [Howard], or as considered here, might seek a realisation using standard sequential systems in MSI form.

Departure from the conventional approach raises fundamental problems, relating to the meaning of automaton "realisation" and the way such a realisation can be formed. The subtlety of the realisation concept has been studied elsewhere [Herman], however the preceeding shows that the basic aim is to provide a translation from input tapes to output codes, in simulation of the objective translation. That is, the realisation must have some state mapping Γ so that $\Gamma_{j_1} \subseteq \alpha * \Gamma \beta^{-1}$, where $\Gamma_{j_1} : X_J^* \longrightarrow Z_J$ is the objective translation, α is an input assignment and β is an output assignment, in other words some circuit state must cover the "root" state j_1. Furthermore an approach for forming such a realisation has been considered, and it has been shown that a realisation can be based on any system of state transitions related by a one-many weak homomorphism to the objective semiautomaton.

It is interesting to consider the conventional procedure for automaton realisation, in the context of this general approach. Using the conventional procedure the designer disregards the objective automaton outputs, and seeks initially to realise the transition aspect, by assigning a state-code to each objective state. The preceeding study shows that the assignment can be an arbitrary one-many weak homomorphism, however unused state-codes become "don't care" conditions in the design of the combinatorial aspect of the realisation, so a one-one assignment is favoured. The desired transition behaviour is then interpreted as a Karnaugh map for each of the state-variables, and this ensures that every transition between objective states is honoured by a transition between the corresponding state-codes. The state-code transitions are otherwise unrestricted, and are set by the use of the "don't care" conditions on the Karnaugh maps for the state-variables. Then the state assignment becomes a one-one partial homomorphism of the objective semiautomaton into the system of state-code transitions, that is the state-assignment becomes a "partial monomorphism". This ensures that the state-code transition system can be used in realising the objective automaton, and it remains to formalise the combinatorial circuitry to generate output codes, in accordance with the requirement $\gamma^{-1}\tilde{x}^{J}\beta \subseteq \widetilde{x)\alpha}^{C}$. It is worth observing that although the state assignment can be an arbitrary partial monomorphism, it is well known that the assignment has a considerable influence on the complexity of the combinatorial circuitry associated with the realisation, and the problem of making an efficient state-assignment has been given extensive consideration [Dolotta & McCluskey; Karp; Armstrong; Hartmanis(b); Stearns & Hartmanis; Curtis].

In assessing stock units in automaton realisation, the problem is to find a one-many weak homomorphism of the objective semiautomaton into some "stock" semi-automaton. Once such a weak homomorphism is found, for example by forming implication trees, the corresponding stock unit can be used in realising the objective automaton. However the designer must be prepared to accept failure, since it may be that no stock semiautomaton is related to the objective semiautomaton by a one-many weak homomorphism. It is then worth forming a reduction of the objective automaton, since the preceeding shows that a stock unit, unsuitable in realising the objective automaton directly, might be useful in realising a reduction. Then if no one-many weak homomorphism can be found the designer can resort to a conventional realisation, since a state-transition system related by a partial monomorphism can easily be formed using bistables and associated combinatorial circuitry. However since no single unit provides a basis for a realisation, the designer might consider whether an interconnection of stock units can be used instead. Thus the designer aims to assess the stock units as components in a "composite" realisation, that is the designer seeks a way of interconnecting several stock units to form a "composite semiautomaton" to serve as the basis of a realisation.

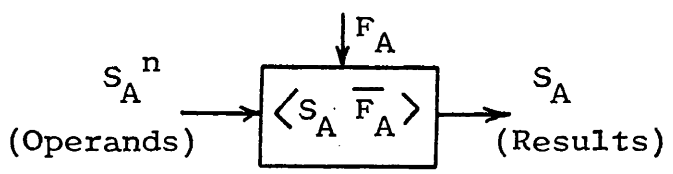
CHAPTER FIVE: Composite Semiautomata

5.1 Introduction

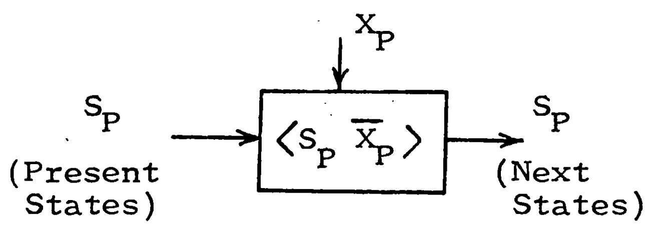
In the case of any F_A-indexed n-ary algebra $\langle S_A \overline{F}_A \rangle$, an arbitrary index $f \in F_A$ defines a corresponding mapping $\overline{f}^A : S_A{}^n \longrightarrow S_A$ as in figure 5.1(a), where $\overline{f}^A \in \overline{F}_A$. Consequently the algebra might be viewed as in figure 5.1(b), where in effect a "result" from S_A is associated with operands from $S_A{}^n$, in accordance with the index from F_A considered. In particular a semiautomaton $P = \langle S_P \overline{X}_P \rangle$ can be regarded as the computation process of figure 5.1(c), a result or "next state" being associated with operands or "present states", in accordance with the input symbol considered.

$$S_A{}^n \xrightarrow{\overline{f}^A} S_A$$

(a) Mapping $\overline{f}^A : S_A{}^n \longrightarrow S_A$



(b) n-ary algebra $\langle S_A \overline{F}_A \rangle$



(c) Semiautomaton $\langle S_P \overline{X}_P \rangle$

Figure 5.1

These visualisations are particularly useful when "component" algebras are combined, and the study will begin by considering just three semiautomata $P = \langle S_P\ \overline{X}_P \rangle$ $Q = \langle S_Q\ \overline{X}_Q \rangle$ and $R = \langle S_R\ \overline{X}_R \rangle$. Each of these semiautomata can be viewed as a separate computation process, however assume a set X is a subset of each of the sets X_P, X_Q and X_R. Then an arbitrary $x \in X$ defines the mappings $\overline{x}^P$, $\overline{x}^Q$ and $\overline{x}^R$ as in figure 5.2(a), and these mappings can be used to define a mapping $\overline{x}^D$ over $S_P \times S_Q \times S_R$.

$$S_P \xrightarrow{\overline{x}^P} S_P \qquad\qquad p \xrightarrow{\overline{x}^P} p'$$

$$S_Q \xrightarrow{\overline{x}^Q} S_Q \qquad\qquad q \xrightarrow{\overline{x}^Q} q'$$

$$S_R \xrightarrow{\overline{x}^R} S_R \qquad\qquad r \xrightarrow{\overline{x}^R} r'$$

(a) Mappings $\overline{x}^P$, $\overline{x}^Q$ and $\overline{x}^R$ associated with $x \in X$

(b) $\langle p\ p' \rangle \in \overline{x}^P$, $\langle q\ q' \rangle \in \overline{x}^Q$ and $\langle r\ r' \rangle \in \overline{x}^R$

$$\begin{tikzcd}
\langle p & q & r \rangle \arrow[r, "\overline{x}^D"] & \langle p' & q' & r' \rangle
\arrow[from=1-1, to=1-4, bend left=50, "\overline{x}^P"]
\arrow[from=1-2, to=1-5, bend left=50, "\overline{x}^Q"]
\arrow[from=1-3, to=1-6, bend left=50, "\overline{x}^R"]
\end{tikzcd}$$

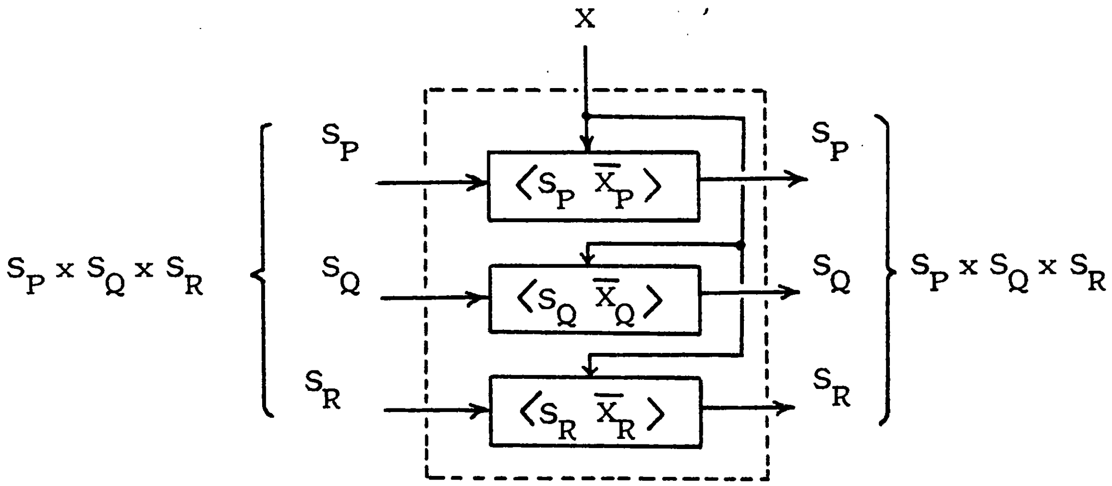
(c) $[\langle p\ q\ r \rangle\ \ \langle p'\ q'\ r' \rangle] \in \overline{x}^D$ iff $\langle p\ p' \rangle \in \overline{x}^P$, $\langle q\ q' \rangle \in \overline{x}^Q$, $\langle r\ r' \rangle \in \overline{x}^R$

Figure 5.2

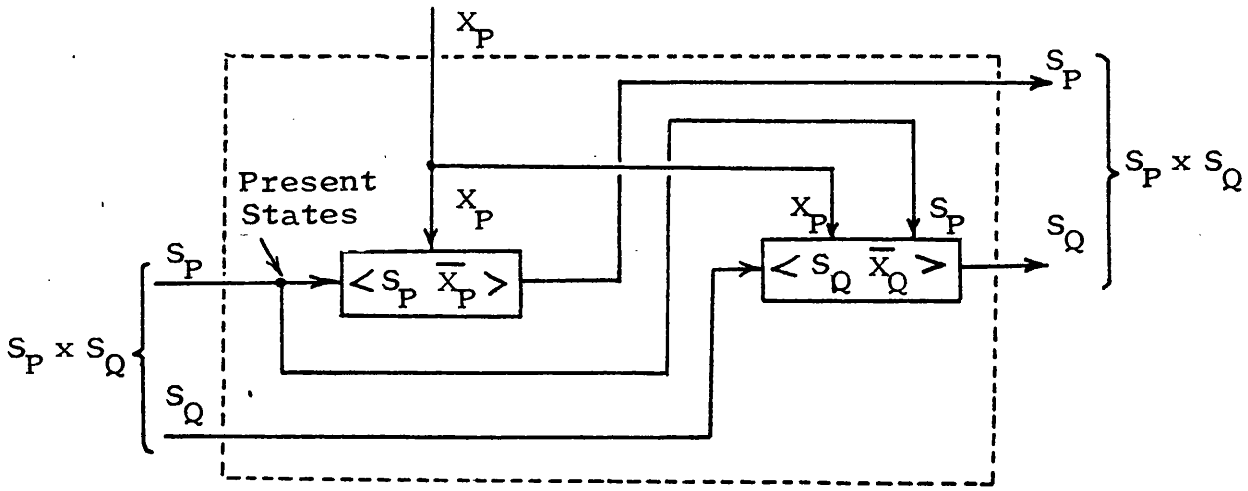
For example figure 5.2(b) shows $\langle p\ p' \rangle \in \overline{x}^P$, $\langle q\ q' \rangle \in \overline{x}^Q$ and $\langle r\ r' \rangle \in \overline{x}^R$, and the mapping $\overline{x}^D$ over $S_P \times S_Q \times S_R$ is defined so that $[\langle p\ q\ r \rangle\ \ \langle p'\ q'\ r' \rangle] \in \overline{x}^D$, as in figure 5.2(c). That is, $[\langle p\ q\ r \rangle\ \ \langle p'\ q'\ r' \rangle] \in \overline{x}^D$ iff $\langle p\ p' \rangle \in \overline{x}^P$, $\langle q\ q' \rangle \in \overline{x}^Q$ and $\langle r\ r' \rangle \in \overline{x}^R$.

Then the combined action of the semiautomata is shown in figure 5.3(a), so that the semiautomata translate present "composite" states from $S_P \times S_Q \times S_R$ into subsequent composite states, and this is achieved by processing each component of the composite state by the appropriate semiautomaton. The composite semiautomaton is then the "direct product" $P \times Q \times R$ of the semiautomata P, Q and R, relative to the set X, and the direct product relative to X can be formalised as the semi-automaton $D = \langle S_D\ \overline{X_D} \rangle$ where $S_D = S_P \times S_Q \times S_R$ and $X_D = X$. Clearly the semiautomaton $D = \langle S_D\ \overline{X_D} \rangle$ is closely related to each of the component semiautomata P, Q, and R, since figure 5.2 shows that a mapping $\overline{x}^D \in \overline{X_D}$ is determined by the associated mappings $\overline{x}^P, \overline{x}^Q$ and $\overline{x}^R$.

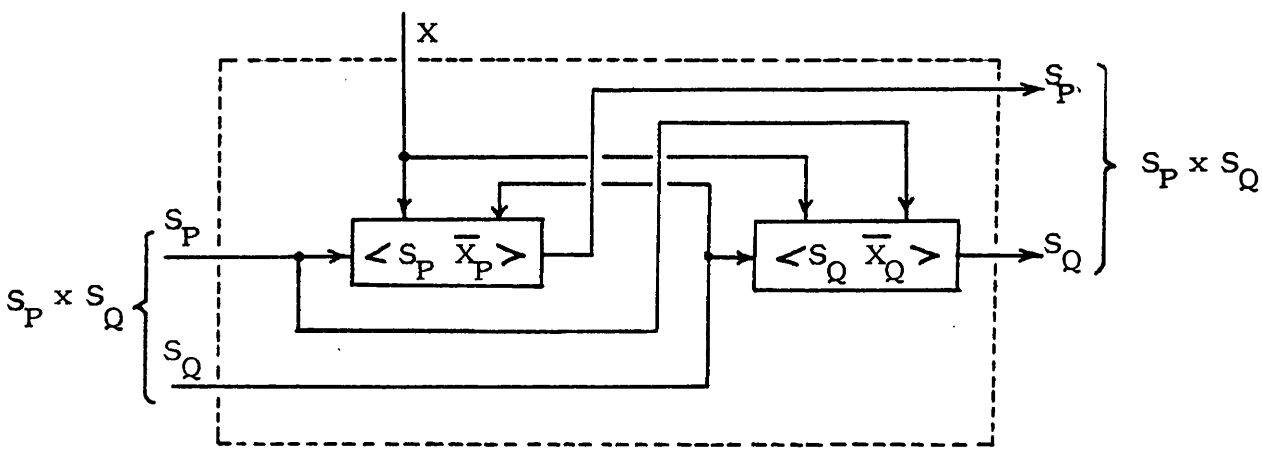
The direct-product construction can be used to express the overall action of several sequential circuits, in particular several sequential MSI units, driven in parallel. However the input sets must be closely related, there must be a subset X common to all the input sets. Alternatively, assume that the semiautomata $P = \langle S_P\ \overline{X_P} \rangle$ and $Q = \langle S_Q\ \overline{X_Q} \rangle$ are such that $X_P \times S_P \subseteq X_Q$.



(a) Composite semiautomaton $D = \langle S_D\ \overline{X}_D \rangle$
where $S_D = S_P \times S_Q \times S_R$ and $X_D = X$



(b) Cascade semiautomaton $P \circ Q$



(c) Feedback interconnection

Figure 5.3

Then the semiautomata P and Q can be "cascaded" as in figure 5.3(b), and this corresponds to supplying both the input code and the present-state code of one sequential circuit as inputs to another. A more complex construction is shown in figure 5.3(c), where $X \times S_Q \subseteq X_P$ and $X \times S_P \subseteq X_Q$. However this "feedback" construction will be of little interest, and attention will be restricted to the cascade and direct-product constructions using partial semiautomata.

5.2 Direct Products

The direct product has been used to form the composite semiautomaton $D = \langle S_D\ \overline{X}_D \rangle$ from component semiautomata P, Q, and R, however the construction is not restricted to semiautomata. In fact a direct product can be formed from any number of algebras of a specific "type" [Gratzer], for example a composite group can be formed from component groups, and a composite ring from component rings.

The present aim is to consider the use of the direct product to combine a number of partial semiautomata, and the "family" concept [Halmos] is invaluable in expressing the general case.

Definition

Let $\{A_i\}_n$ be a family, with index set $n = \{0, 1, 2, \ldots . n-1\}$, of semiautomata $A_i = \langle S_i\ \overline{X}_i \rangle$ and let X be a set where $(\forall i)(i \in n \Longrightarrow X \subseteq X_i)$.

The <u>Direct Product</u> $\Pi\{A_i\}_n$ relative to set X is the semiautomaton D = $\langle S_D\ \overline{X}_D \rangle$ where X_D = X, $S_D = X\{S_i\}_n$ and $(\forall x)(x \in X \Longrightarrow \overline{x}^D \in \overline{X}_D)$ where

$$\overline{x}^D = \left\{ [\langle a_0 \ldots a_{n-1}\rangle\ \langle a'_0 \ldots a'_{n-1}\rangle] \;\middle|\; \begin{array}{l} \langle a_0 \ldots a_{n-1}\rangle \in X\{S_i\}_n, \\ \langle a'_0 \ldots a'_{n-1}\rangle \in X\{S_i\}_n \\ \& \ (\forall i)(i \in n \Longrightarrow \langle a_i\ a'_i\rangle) \in \overline{x}^i \end{array} \right\}$$

Consider for example the family $\{A_i\}_n$ where n = {0, 1, 2}, in which case the semiautomata are $A_0 = \langle S_0\ \overline{X}_0\rangle$, $A_1 = \langle S_1\ \overline{X}_1\rangle$ and $A_2 = \langle S_2\ \overline{X}_2\rangle$. Assuming the set X to be a subset of each of the sets X_0, X_1, X_2 then $x \in X$ implies $\overline{x}^0 \in \overline{X}_0$, $\overline{x}^1 \in \overline{X}_1$ and $\overline{x}^2 \in \overline{X}_2$, and these mappings are used to define a mapping $\overline{x}^D$ over $X\{S_i\}_n = S_0 \times S_1 \times S_2$ by setting

$$\overline{x}^D = \left\{ [\langle a_0 a_1 a_2\rangle\ \langle a'_0 a'_1 a'_2\rangle] \;\middle|\; \begin{array}{l} \langle a_0 a_1 a_2\rangle, \langle a'_0 a'_1 a'_2\rangle \in S_0 \times S_1 \times S_2, \\ \langle a_0\ a'_0\rangle \in \overline{x}^0,\ \langle a_1\ a'_1\rangle \in \overline{x}^1, \\ \&\ \ \langle a_2\ a'_2\rangle \in \overline{x}^2 \end{array} \right\}$$

Defining A_0 = P, A_1 = Q and A_2 = R, the direct product $\Pi\{A_i\}_n$ is then the semiautomaton D = $\langle S_D\ \overline{X}_D\rangle$ of figure 5.3(a).

It is especially important to note that semiautomaton D = P x Q x R is completely determined by the component semiautomata and the choice of the set X, so there is a close relationship between D and each of the semiautomata P, Q and R. For example the "projection" $\emptyset_P$ from $S_P \times S_Q \times S_R$ to S_P, where

$$\emptyset_P = \{[\langle p\ q\ r\rangle\ \ p] \mid \langle p\ q\ r\rangle \in S_P \times S_Q \times S_R\},$$

establishes the natural relationship between D and semiautomaton P by isolating the S_P component of each composite state. Clearly $\emptyset_P$ is a mapping with $S_P \times S_Q \times S_R$ as domain, since each element from $S_P \times S_Q \times S_R$ involves a particular component from S_P. Indeed, $\emptyset_P$ is a partial homomorphism of the composite semiautomaton $P \times Q \times R$ to the component semiautomaton P. To confirm this assume $x \in X$, and assume $[\langle p\ q\ r\rangle\ \ p'] \in \overline{x}^D \emptyset_P$. Then $[\langle p\ q\ r\rangle\ \ \langle p'\ q'\ r'\rangle] \in \overline{x}^D$ and $[\langle p'\ q'\ r'\rangle\ \ p'] \in \emptyset_P$ for some $\langle p'\ q'\ r'\rangle$, furthermore $[\langle p\ q\ r\rangle\ \ \langle p'\ q'\ r'\rangle] \in \overline{x}^D$ implies $\langle p\ p'\rangle \in \overline{x}^P$, $\langle q\ q'\rangle \in \overline{x}^Q$ and $\langle r\ r'\rangle \in \overline{x}^R$. Clearly $[\langle p\ q\ r\rangle\ \ p] \in \emptyset_P$, and $\langle p\ p'\rangle \in \overline{x}^P$ so $[\langle p\ q\ r\rangle\ \ p'] \in \emptyset_P \overline{x}^P$. Hence $[\langle p\ q\ r\rangle\ \ p'] \in \overline{x}^D \emptyset_P$ implies $[\langle p\ q\ r\rangle\ \ p'] \in \emptyset_P \overline{x}^P$, and $x \in X$ is arbitrary so $(\forall x)(x \in X \Longrightarrow \overline{x}^D \emptyset_P \subseteq \emptyset_P \overline{x}^P)$. Hence $\emptyset_P$ is a partial homomorphism of $P \times Q \times R$ to P, indeed $\emptyset_P$ is a partial epimorphism of $P \times Q \times R$ "onto" P, since $\emptyset_P$ has S_P as codomain so long as S_P, S_Q and S_R are nonvoid. Similarly the projection

$$\emptyset_Q = \left\{ [\langle p\ q\ r\rangle\ \ q] \ \middle|\ \langle p\ q\ r\rangle \in S_P \times S_Q \times S_R \right\}$$

is a partial epimorphism of $P \times Q \times R$ onto Q, and the projection

$$\emptyset_R = \left\{ [\langle p\ q\ r\rangle\ \ r] \ \middle|\ \langle p\ q\ r\rangle \in S_P \times S_Q \times S_R \right\}$$

is a partial epimorphism of $P \times Q \times R$ onto R.

The projections are particularly useful when an automaton is realised in the form of a direct product.

For example suppose an objective automaton $\hat{J}$ is realised by forming a reduction $\hat{\mathfrak{J}}$, and finding a one-many weak homomorphism γ of semiautomaton $\mathfrak{J}$ into the direct product $D = P \times Q \times R$, so that D can be used to form an automaton $\hat{D}$ where $\hat{\mathfrak{J}} \leqslant \hat{D}$. The canonical relation π from J to $\mathfrak{J}$ is of course a weak homomorphism, since reduction $\hat{\mathfrak{J}}$ is based on an image of the objective semiautomaton J, and the projections $\emptyset_P$, $\emptyset_Q$ and $\emptyset_R$ are partial homomorphisms so they are weak homomorphisms. These relationships are shown in the "weak-homomorphism diagram" of figure 5.4, and the diagram is readily extended since a product of weak homomorphisms is a weak homomorphism.

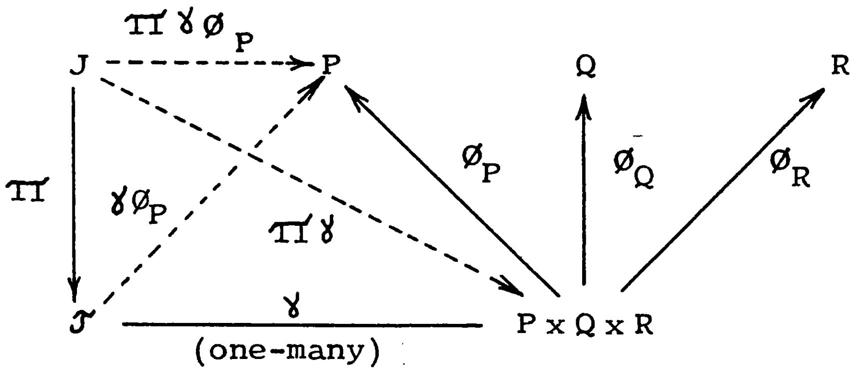


Figure 5.4 Weak-Homomorphism diagram

For example $J \xrightarrow{\pi} \mathfrak{J}$ and $\mathfrak{J} \xrightarrow{\gamma} P \times Q \times R$ so $J \xrightarrow{\pi\gamma} P \times Q \times R$, showing that the objective semiautomaton is related to the composite semiautomaton by a weak homomorphism, and similarly $\gamma\emptyset_P$ is a weak homomorphism of $\mathfrak{J}$ to component semiautomaton P.

Furthermore $J \xrightarrow{\pi'} \mathcal{J}$, $\mathcal{J} \xrightarrow{\gamma} P \times Q \times R$ and $P \times Q \times R \xrightarrow{\emptyset_P} P$ so $J \xrightarrow{\pi' \gamma \emptyset_P} P$, and this weak homomorphism is particularly illustrative. Assuming $\langle j\ j' \rangle \in \bar{x}^J$ and $\langle j\ p \rangle \in \pi' \gamma \emptyset_P$, then $\langle j\ p \rangle \in \pi' \gamma \emptyset_P$ implies $\langle j\ B \rangle \in \pi'$, $[B\ \ \langle p\ q\ r \rangle] \in \gamma$ and $[\langle p\ q\ r \rangle\ \ p] \in \emptyset_P$ for some $B \in S_{\mathcal{J}}$ and some $\langle p\ q\ r \rangle \in S_P \times S_Q \times S_R$, as shown in figure 5.5.

$$\begin{tikzcd}
j \arrow[r, "\pi'\gamma\emptyset_P"] \arrow[d, "\bar{x}^J"'] & p \\
j' &
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
j \arrow[r, "\pi'"] \arrow[d, "\bar{x}^J"] & B \arrow[r, "\gamma"] \arrow[d, "\bar{x}^{\mathcal{J}}"] & \langle p\ q\ r \rangle \arrow[r, "\emptyset_P"] \arrow[d, "\bar{x}^D"] & p \arrow[d, "\bar{x}^P"] \\
j' \arrow[r, "\pi'"] & B' \arrow[r, "\gamma"] & \langle p'\ q'\ r' \rangle \arrow[r, "\emptyset_P"] & p'
\end{tikzcd}$$

<u>Figure 5.5</u> $\underline{J \xrightarrow{\pi'\gamma\emptyset_P} P}$

Then $\langle B\ j' \rangle \in \pi'^{-1} \bar{x}^J$, and π' is a weak homomorphism so $\pi'^{-1} \bar{x}^J \subseteq \bar{x}^{\mathcal{J}} \pi'^{-1}$, giving $\langle B\ j' \rangle \in \bar{x}^{\mathcal{J}} \pi'^{-1}$ so $\langle B\ B' \rangle \in \bar{x}^{\mathcal{J}}$ and $\langle j'\ B' \rangle \in \pi'$ for some $B' \in S_{\mathcal{J}}$, as shown. Then $[\langle p\ q\ r \rangle\ \ B'] \in \gamma^{-1} \bar{x}^{\mathcal{J}}$ where $\mathcal{J} \xrightarrow{\gamma} P \times Q \times R$, so $[\langle p\ q\ r \rangle\ \ B'] \in \bar{x}^D \gamma^{-1}$, in which case $[\langle p\ q\ r \rangle\ \ \langle p'\ q'\ r' \rangle] \in \bar{x}^D$ and $[B'\ \ \langle p'\ q'\ r' \rangle] \in \gamma$ for some $\langle p'\ q'\ r' \rangle$. Furthermore $P \times Q \times R \xrightarrow{\emptyset_P} P$, so $\langle p\ p' \rangle \in \bar{x}^P$ and $[\langle p'\ q'\ r' \rangle\ \ p'] \in \emptyset_P$, in which case $\langle j'\ p' \rangle \in \pi' \gamma \emptyset_P$. In effect the association $\langle j\ j' \rangle \in \bar{x}^J$ is "followed" by the association $\langle B\ B' \rangle \in \bar{x}^{\mathcal{J}}$ since $J \xrightarrow{\pi'} \mathcal{J}$, the association

$\langle B\ B' \rangle \ \epsilon\ \overline{x}^{\mathcal{J}}$ is followed by the association $[\langle p\ q\ r \rangle\ \langle p'\ q'\ r' \rangle]\ \epsilon\ \overline{x}^{D}$ since $\mathcal{J} \xrightarrow{\gamma} P \times Q \times R$, and then $\langle p\ p' \rangle\ \epsilon\ \overline{x}^{P}$ since $P \times Q \times R \xrightarrow{\phi_P} P$. This illustrates that $\Pi\gamma\phi_P$ is a weak homomorphism of J to the component semiautomaton P, and similarly $J \xrightarrow{\Pi\gamma\phi_Q} Q$ and $J \xrightarrow{\Pi\gamma\phi_R} R$.

The projections ϕ_P, ϕ_Q, ϕ_R are weak homomorphisms relating direct product $P \times Q \times R$ to the component semiautomata, however there is an interesting generalisation. In figure 5.3(a) the semiautomata P, Q, R form the direct product $D = P \times Q \times R$, and any two component semiautomata form a composite semiautomaton "within" $P \times Q \times R$. For example, the semiautomata P and Q form a direct product $E = P \times Q$ such that $E = \langle S_E\ \overline{X}_E \rangle$, $S_E = S_P \times S_Q$ and $(\forall x)(x \in X \Longrightarrow \overline{x}^E \in \overline{X}_E)$ where

$$\overline{x}^E = \left\{ [\langle p\ q \rangle\ \langle p'\ q' \rangle] \;\middle|\; \begin{array}{l} \langle p\ q \rangle, \langle p'\ q' \rangle \ \epsilon\ S_P \times S_Q, \\ \langle p\ p' \rangle \ \epsilon\ \overline{x}^P \ \&\ \langle q\ q' \rangle \ \epsilon\ \overline{x}^Q \end{array} \right\}$$

Then the projection

$$\phi_{PQ} = \left\{ [\langle p\ q\ r \rangle\ \langle p\ q \rangle] \;\middle|\; \langle p\ q\ r \rangle \ \epsilon\ S_P \times S_Q \times S_R \right\}$$

is a partial epimorphism of $P \times Q \times R$ onto $P \times Q$, and in general a direct product "within" another will always be an image under a partial epimorphism. To confirm ϕ_{PQ} to be a partial homomorphism assume $x \in X$, and assume

$[\langle p\ q\ r \rangle\ \langle p'\ q' \rangle]\ \epsilon\ \overline{x}^D\ \phi_{PQ}$. Then

$[\langle p\ q\ r \rangle\ \langle p'\ q'\ r' \rangle]\ \epsilon\ \overline{x}^D$ and

$[\langle p'\ q'\ r' \rangle\ \langle p'\ q' \rangle]\ \epsilon\ \phi_{PQ}$ for some

$\langle p'\ q'\ r'\rangle \in S_P \times S_Q \times S_R$, as shown in figure 5.6, and $[\langle p\ q\ r\rangle\ \langle p'\ q'\ r'\rangle] \in \overline{x}^D$ implies $\langle p\ p'\rangle \in \overline{x}^P$, $\langle q\ q'\rangle \in \overline{x}^Q$ and $\langle r\ r'\rangle \in \overline{x}^R$. Furthermore $\langle p\ p'\rangle \in \overline{x}^P$, $\langle q\ q'\rangle \in \overline{x}^Q$ implies $[\langle p\ q\rangle\ \langle p'\ q'\rangle] \in \overline{x}^E$, and $[\langle p\ q\ r\rangle\ \langle p\ q\rangle] \in \emptyset_{PQ}$ so $[\langle p\ q\ r\rangle\ \langle p'\ q'\rangle] \in \emptyset_{PQ}\,\overline{x}^E$, as shown.

$$\begin{tikzcd}
\langle p\ q\ r\rangle \arrow[r, "\emptyset_{PQ}"] \arrow[d, "\overline{x}^D"'] & \langle p\ q\rangle \arrow[d, "\overline{x}^E"] \\
\langle p'\ q'\ r'\rangle \arrow[r, "\emptyset_{PQ}"] & \langle p'\ q'\rangle
\end{tikzcd}$$

<u>Figure 5.6</u> <u>$P \times Q \times R \xrightarrow{\emptyset_{PQ}} P \times Q$</u>

Hence $[\langle p\ q\ r\rangle\ \langle p'\ q'\rangle] \in \overline{x}^D\,\emptyset_{PQ}$ implies $[\langle p\ q\ r\rangle\ \langle p'\ q'\rangle] \in \emptyset_{PQ}\,\overline{x}^E$, and $x \in X$ is arbitrary so $(\forall x)(x \in X \Longrightarrow \overline{x}^D\,\emptyset_{PQ} \subseteq \emptyset_{PQ}\,\overline{x}^E)$. Finally $\emptyset_{PQ}$ has $S_P \times S_Q \times S_R$ as domain, and has $S_P \times S_Q$ as codomain, so $\emptyset_{PQ}$ is a partial epimorphism of $P \times Q \times R$ onto $P \times Q$.

The family concept is again important in considering the general case. By a direct product "within" a direct product $\Pi\{A_i\}_n$ is meant a direct product $\Pi\{A_i\}_\Theta$ where $\Theta \subseteq n$, and then the projection

$$\emptyset_\Theta = \{\langle a\ \alpha\rangle \mid a \in X\{S_i\}_n,\ \alpha \in X\{S_i\}_\Theta\ \ \&\ (\forall i)(i \in \Theta \Rightarrow a_i = \alpha_i)\}$$

is a partial epimorphism of $\Pi\{A_i\}_n$ onto $\Pi\{A_i\}_\Theta$

Theorem

Projection $\emptyset_\Theta$ is a partial epimorphism of $\Pi\{A_i\}_n$ onto $\Pi\{A_i\}_\Theta$, for any $\Theta \subsetneq n$.

Proof

Assume $\Theta \subsetneq n$, define $\Pi\{A_i\}_n = D = \langle S_D\ \overline{X}_D\rangle$, define $\Pi\{A_i\}_\Theta = E = \langle S_E\ \overline{X}_E\rangle$ and define $X_D = X_E = X$. Then $x \in X$ implies $\overline{x}^D \in \overline{X}_D$ where

$$\overline{x}^D = \{\langle a\ a'\rangle \mid a,a' \in X\{S_i\}_n \ \&\ (\forall i)(i \in n \Rightarrow \langle a_i\ a'_i\rangle \in \overline{x}^i)\},$$

furthermore $x \in X$ implies $\overline{x}^E \in \overline{X}_E$ where

$$\overline{x}^E = \{\langle \alpha\ \alpha'\rangle \mid \alpha,\alpha' \in X\{S_i\}_\Theta \ \&\ (\forall i)(i \in \Theta \Rightarrow \langle \alpha_i\ \alpha'_i\rangle \in \overline{x}^i)\}$$

Assume $x \in X$, and assume $\langle a\ \alpha'\rangle \in \overline{x}^D \emptyset_\Theta$. Clearly $\langle a\ \alpha\rangle \in \emptyset_\Theta$ for some α where $(\forall i)(i \in \Theta \Rightarrow a_i = \alpha_i)$, furthermore $\langle a\ \alpha'\rangle \in \overline{x}^D \emptyset_\Theta$ implies $\langle a\ a'\rangle \in \overline{x}^D$ and $\langle a'\ \alpha'\rangle \in \emptyset_\Theta$ for some a', and then $\langle a'\ \alpha'\rangle \in \emptyset_\Theta$ implies $(\forall i)(i \in \Theta \Rightarrow a'_i = \alpha'_i)$.

In addition $\langle a\ a'\rangle \in \overline{x}^D$ implies $(\forall i)(i \in n \Rightarrow \langle a_i\ a'_i\rangle \in \overline{x}^i)$, and $\Theta \subsetneq n$ so $(\forall i)(i \in \Theta \Rightarrow \langle a_i\ a'_i\rangle \in \overline{x}^i)$. Assuming $i \in \Theta$ then $\langle a_i\ a'_i\rangle \in \overline{x}^i$, and from above $a_i = \alpha_i$ and $a'_i = \alpha'_i$ so $\langle \alpha_i\ \alpha'_i\rangle \in \overline{x}^i$. Hence $(\forall i)(i \in \Theta \Rightarrow \langle \alpha_i\ \alpha'_i\rangle \in \overline{x}^i)$, in which case $\langle \alpha\ \alpha'\rangle \in \overline{x}^E$, and $\langle a\ \alpha\rangle \in \emptyset_\Theta$ so $\langle a\ \alpha'\rangle \in \emptyset_\Theta\, \overline{x}^E$.

Therefore $\langle a\ \alpha'\rangle \in \overline{x}^D \emptyset_\Theta$ implies $\langle a\ \alpha'\rangle \in \emptyset_\Theta\, \overline{x}^E$, and $x \in X$ is arbitrary so $(\forall x)(x \in X \Rightarrow \overline{x}^D \emptyset_\Theta \subsetneq \emptyset_\Theta\, \overline{x}^E)$. Considering now the nature of $\emptyset_\theta$ clearly $\emptyset_\theta$ is a mapping with domain $X\{S_i\}_n$, since any element from $X\{S_i\}_n$ involves particular terms indexed by $\theta \subseteq n$, and these terms form a particular element of $X\{S_i\}_\theta$. Furthermore $\emptyset_\theta$ has codomain $X\{S_i\}_\theta$ so long as the sets from the family $\{S_i\}_n$ are nonvoid, and this confirms projection $\emptyset_\theta$ to be a partial epimorphism of $\Pi\{A_i\}_n$ onto $\Pi\{A_i\}_\theta$, completing the proof.

This shows that the projection of a direct product onto an "internal" direct product is a partial homomorphism, and it is useful to observe that a family $\{A_i\}_n$ of complete semiautomata will define a complete composite semiautomaton $\Pi\{A_i\}_n$, so a projection $\emptyset_\theta$ will be a homomorphism. For example if a number of MSI sequential units are driven in parallel, the overall action can be represented as a complete semiautomaton in the form of a direct product. Then any collection of the component units forms an internal direct product, and the associated projection is an epimorphism.

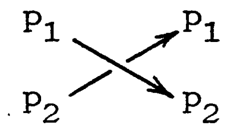
It is important also to consider the case of a family $\{A_i\}_\theta = \{A_O\}$, that is where the indexed set within a direct product $\Pi\{A_i\}_n$ of partial semiautomata comprises just one semiautomaton $A_O = \langle S_O \overline{X}_O \rangle$. The direct product $\Pi\{A_i\}_\theta$ can be replaced by the semiautomaton A_O, and the projection $\emptyset_\theta$ can be replaced by a projection $\emptyset_O$ from $X\{S_i\}_n$ to S_O. Then $\emptyset_O$ is a partial epimorphism of $\Pi\{A_i\}_n$ onto A_O, just as the projection $\emptyset_P$ is a partial epimorphism of $P \times Q \times R$ onto P, and similarly for the projections $\emptyset_Q$ and $\emptyset_R$.

5.3 Cascade Products

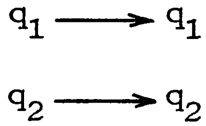
The "cascade product" $P \circ Q$ of a semiautomaton $P = \langle S_P\ \overline{X}_P \rangle$ and a semiautomaton $Q = \langle S_Q\ \overline{X}_Q \rangle$, assuming $X_P \times S_P \subseteq X_Q$, has been illustrated in figure 5.3(b), and the composite semiautomaton can be formalised as $C = \langle S_C\ \overline{X}_C \rangle$ where $S_C = S_P \times S_Q$ and $X_C = X_P$. The semiautomaton represents the circuit formed by feeding the input code and present-state code of one sequential circuit as inputs to another, so that the state transitions of the "dependent" circuit are influenced by the present state of the "independent" circuit.

To formalise the cascade construction assume $S_P = \{p_1, p_2\}$ and $S_Q = \{q_1, q_2\}$, assume $x \in X_P$ and let the mapping $\overline{x}^P$ associated with semiautomaton P be that of figure 5.7(a). Then $\langle x\ p_1 \rangle$, $\langle x\ p_2 \rangle \in X_P \times S_P$, and semiautomaton $Q = \langle S_Q\ \overline{X}_Q \rangle$ is indexed by X_Q where $X_P \times S_P \subseteq X_Q$, so each element of $X_P \times S_P$ defines an associated mapping over S_Q. In particular $\langle x\ p_1 \rangle \in X_P \times S_P$ indexes a mapping $\overline{\langle x\ p_1 \rangle}^Q$ over S_Q such as that of figure (b), and $\langle x\ p_2 \rangle$ indexes a mapping $\overline{\langle x\ p_2 \rangle}^Q$ such as that of figure (c).

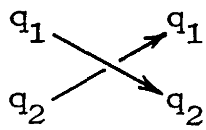
Then the mappings $\overline{x}^P$, $\overline{\langle x\ p_1 \rangle}^Q$ and $\overline{\langle x\ p_2 \rangle}^Q$ combine to form the mapping $\overline{x}^C$ over $S_P \times S_Q$, as in figure 5.7(d). For example consider the composite state



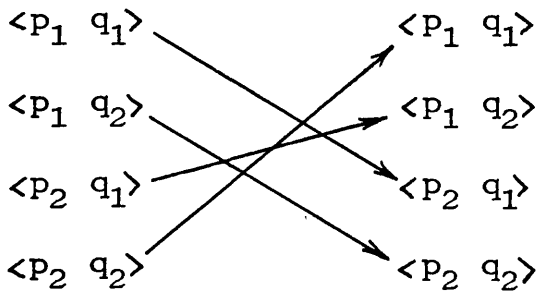
(a) $\bar{x}^P : S_P \longrightarrow S_P$



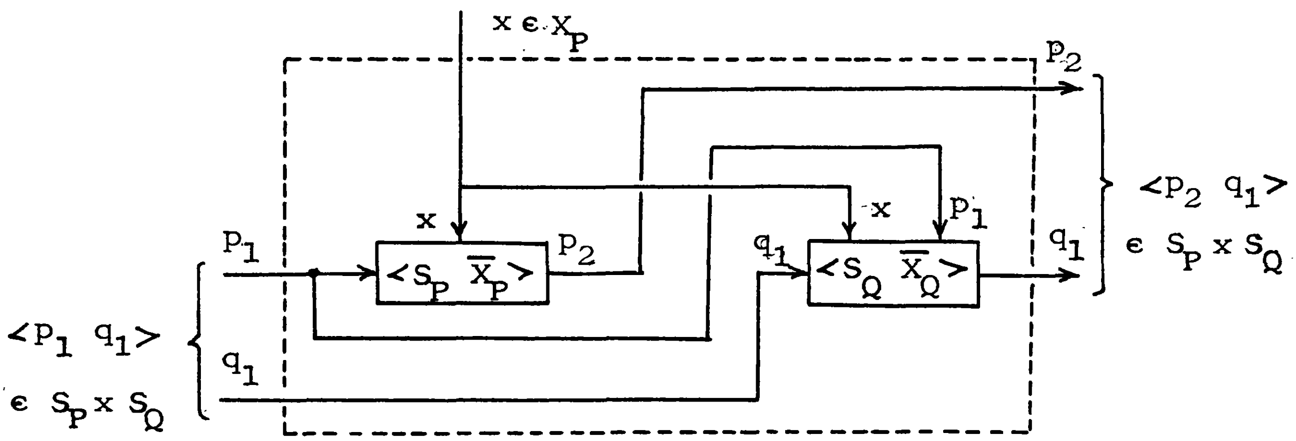
(b) $\overline{\langle x\ p_1\rangle}^Q : S_Q \longrightarrow S_Q$



(c) $\overline{\langle x\ p_2\rangle}^Q : S_Q \longrightarrow S_Q$



(d) $\bar{x}^C : S_P \times S_Q \longrightarrow S_P \times S_Q$



(e) $[\langle p_1\ q_1\rangle\ \ \langle p_2\ q_1\rangle] \in \bar{x}^C$ since $\langle p_1\ p_2\rangle \in \bar{x}^P$

& $\langle q_1\ q_1\rangle \in \overline{\langle x\ p_1\rangle}^Q$

Figure 5.7

$\langle p_1\ q_1\rangle\ \epsilon\ S_P \times S_Q$, as shown in figure 5.7(e). The $\bar{x}^C$-successor of $\langle p_1\ q_1\rangle$ is derived by relating the S_P term to semiautomaton P and the S_Q term to semiautomaton Q, and figure 5.7(a) shows p_2 to be the $\bar{x}^P$-successor of p_1 so the $\bar{x}^C$-successor of composite state $\langle p_1\ q_1\rangle$ will have p_2 as the S_P component. The S_Q component is obtained by considering semiautomaton Q, and using the input symbol x and the state symbol p_1 to form an index $\langle x\ p_1\rangle\ \epsilon\ X_Q$. Then from figure 5.7(b) the $\overline{\langle x\ p_1\rangle}^Q$-successor of q_1 is q_1, so the $\bar{x}^C$-successor of the composite state $\langle p_1\ q_1\rangle$ has S_Q component q_1. Consequently $\langle p_1\ q_1\rangle$ has $\langle p_2\ q_1\rangle$ as $\bar{x}^C$-successor, that is $[\langle p_1\ q_1\rangle\ \ \langle p_2\ q_1\rangle]\ \epsilon\ \bar{x}^C$, and repeating this argument to find the $\bar{x}^C$-successor for each composite state gives the mapping $\bar{x}^C$ of figure 5.7(d).

More formally, the mapping $\bar{x}^C$ over $S_P \times S_Q$ is defined as

$$\bar{x}^C = \left\{ [\langle p\ q\rangle\ \ \langle p'\ q'\rangle] \;\middle|\; \begin{array}{l} \langle p\ q\rangle\ ,\ \langle p'\ q'\rangle\ \epsilon\ S_P \times S_Q, \\ \langle p\ p'\rangle\ \ \epsilon\ \bar{x}^P\ \ \& \\ \langle q\ q'\rangle\ \ \epsilon\ \overline{\langle x\ p\rangle}^Q \end{array} \right\}$$

that is $[\langle p\ q\rangle\ \ \langle p'\ q'\rangle]\ \epsilon\ \bar{x}^C$ iff

$$\langle p\ q\rangle\ ,\ \langle p'\ q'\rangle\ \ \epsilon\ S_P \times S_Q$$

where p' is the $\bar{x}^P$-successor of p and q' is the $\overline{\langle x\ p\rangle}^Q$-successor of q. In fact the cascade construction can be used to combine any number of suitable semiautomata, so that a family $\{A_i\}_n$ can be used to form a cascade $C\{A_i\}_n$.

Definition

Let $\{A_i\}_n$ be a family, with index set $n = \{0, 1, 2, \ldots n-1\}$, of semiautomata $A_i = \langle S_i \ \overline{X}_i \rangle$ such that

$$(\forall i)(i \in n,\ i \neq \emptyset \Longrightarrow X_0 \times S_{i-1} \subseteq X_i)$$

The Cascade Product $C\{A_i\}_n$ over the family is the semiautomaton $C = \langle S_C \ \overline{X}_C \rangle$ such that $X_C = X_0$, $S_C = X\{S_i\}_n$ and $(\forall x)(x \in X_0 \Longrightarrow \overline{x}^C \in \overline{X}_C)$ where

$$\overline{x}^C = \left\{ \langle a \ a' \rangle \;\middle|\; \begin{array}{l} a, a' \in X\{S_i\}_n,\ \langle a_0 \ a'_0 \rangle \in \overline{x}^0 \quad \& \\ (\forall i)(i \in n,\ i \neq \emptyset \Longrightarrow \langle a_i \ a'_i \rangle \in \overline{\langle x \ a_{i-1} \rangle}^{\,i}\ ) \end{array} \right\}$$

It was shown that each component semiautomaton of a direct product is an image of the direct product under a partial epimorphism, however the component algebras in a cascade product are not closely related to the composite semiautomaton. The composite semiautomaton has input set X_0, and an arbitrary component semiautomaton A_i has input set X_i where $X_0 \times S_{i-1} \subseteq X_i$, so the component and composite semiautomata are differently indexed. The exception is the very first semiautomaton in a cascade product, and the associated projection is then a partial epimorphism.

Theorem

The projection $\emptyset_0 = \{\langle a \ \alpha \rangle \mid a \in X\{S_i\}_n,\ \alpha = a_0\}$ of $X\{S_i\}_n$ onto S_0 is a partial epimorphism of $C\{A_i\}_n$ onto A_0.

Proof

Assume $x \in X_0$, and assume $\langle a\ \alpha' \rangle \in \bar{x}^C \phi_0$ so $\langle a\ a' \rangle \in \bar{x}^C$ and $\langle a'\ \alpha' \rangle \in \phi_0$ for some a'. Then $\alpha' = a'_0$, and it is evident that $\langle a\ \alpha \rangle \in \phi_0$ for some α where $\alpha = a_0$. In addition $\langle a\ a' \rangle \in \bar{x}^C$ implies $\langle a_0\ a'_0 \rangle \in \bar{x}^0$, and $\alpha = a_0$, $\alpha' = a'_0$ so $\langle \alpha\ \alpha' \rangle \in \bar{x}^0$. From above $\langle a\ \alpha \rangle \in \phi_0$, therefore $\langle a\ \alpha' \rangle \in \phi_0 \bar{x}^0$.

Consequently $\langle a\ \alpha' \rangle \in \bar{x}^C \phi_0$ implies $\langle a\ \alpha' \rangle \in \phi_0 \bar{x}^0$, and $x \in X$ is arbitrary so $(\forall x)(x \in X_0 \Longrightarrow \bar{x}^C \phi_0 \subseteq \phi_0 \bar{x}^0)$. Furthermore ϕ_0 is a mapping of $X\{S_i\}_n$ onto S_0, so ϕ_0 is a partial epimorphism.

The cascade product is of particular importance when combined with the direct product, for example a direct product $P \times Q$ can be cascaded with an appropriate semiautomaton R to form a composite semiautomaton $C = (P \times Q) \circ R$. The associated partial homomorphism diagram is that of figure 5.8, and shows that projection ϕ_{PQ} is a partial homomorphism of the cascade semiautomaton $C = (P \times Q) \circ R$ onto the first semiautomaton in the chain, where here this is the direct product $P \times Q$.

$$
\begin{tikzcd}
 & P \\
C \arrow[ur, "\phi_{PQ}\phi_P"] \arrow[r, "\phi_{PQ}"'] \arrow[dr, "\phi_{PQ}\phi_Q"'] & P \times Q \arrow[u, "\phi_P"'] \arrow[d, "\phi_Q"] \\
 & Q
\end{tikzcd}
$$

Figure 5.8 Partial-homomorphism diagram

Furthermore the projection $\emptyset_P$ is a partial homomorphism of $P \times Q$ onto P, so $\emptyset_{PQ}\,\emptyset_P$ is a partial homomorphism of cascade semiautomaton C onto P, and similarly for $\emptyset_{PQ}\,\emptyset_Q$. The semiautomaton R is unrelated to the composite semiautomaton $C = (P \times Q) \circ R$ and cannot be represented on the diagram, however semiautomaton R is related to $P \times Q$, and this could be expressed as a partition pair [Hartmanis & Stearns].

5.4 Conclusion

A circuit composed of MSI (Medium Scale Integration) units can be represented as a composite semiautomaton, and can be analysed in the normal way, however the analysis of such a composite circuit introduces additional problems. It is important to be able to express the way each part of the circuit is influenced by other parts, and this can be expressed using appropriate partition pairs. Furthermore, it is important to be able to express the way each part of the circuit contributes to the overall circuit action.

In the case of a number of MSI units driven in parallel, the relationship between the composite and component semiautomata is readily expressed using the homomorphism concept. The preceeding shows that each component semiautomaton is an image of the composite semiautomaton under a homomorphism, so that each MSI unit represents a homomorphic image of the system of composite-state transitions. More specifically let semiautomata $P = \langle S_P\ \overline{X}_P \rangle$ and $Q = \langle S_Q\ \overline{X}_Q \rangle$ represent MSI units, so that $P \times Q$ represents the circuit formed by driving the MSI units in parallel, assume $S_P = \{p_1, p_2\}$ and assume $S_Q = \{q_1, q_2\}$. Then the projection $\emptyset_P$ is a homomorphism of $P \times Q$ onto P, and each state from S_P represents an "ambiguity" regarding the state of the composite circuit. For example p_1 represents the set of the composite states with p_1 as the S_P component, that is p_1 represents the ambiguity $\{\langle p_1\ q_1\rangle\ \langle p_1\ q_2\rangle\}$. This expresses that observing the circuit represented as P to be in state p_1 does not reveal the exact state of the composite circuit represented as $P \times Q$, but reveals instead that the composite state must be either $\langle p_1\ q_1\rangle$ or $\langle p_1\ q_2\rangle$. Similarly the state $p_2 \in S_P$ represents the ambiguity $\{\langle p_2\ q_1\rangle\ \langle p_2\ q_2\rangle\}$, the state $q_1 \in S_Q$ represents the ambiguity $\{\langle p_1\ q_1\rangle\ \langle p_2\ q_1\rangle\}$ and state $q_2 \in S_Q$ represents the ambiguity $\{\langle p_1\ q_2\rangle\ \langle p_2\ q_2\rangle\}$, where each of these ambiguities is a subset of $S_P \times S_Q$.

The ambiguities are closely related to the homomorphisms $\emptyset_P$ and $\emptyset_Q$, and any homomorphism can be interpreted in terms of ambiguities regarding the parent algebra. However the homomorphic images P and Q of composite algebra $P \times Q$ are particularly important "mutually resolving" homomorphic images, since intersecting the ambiguities produces singleton or "resolved" ambiguities. For example state $p_1 \in S_P$ represents the ambiguity $\{\langle p_1\ q_1\rangle\ \langle p_1\ q_2\rangle\}$ and state q_1 represents the ambiguity $\{\langle p_1\ q_1\rangle\ \langle p_2\ q_1\rangle\}$, so the composite state $\langle p_1\ q_1\rangle$ represents the intersection of these ambiguities, that is $\langle p_1\ q_1\rangle$ represents the "resolved" ambiguity $\{\langle p_1\ q_1\rangle\}$. It is easily verified that the states from S_P and S_Q represent mutually resolving ambiguities, so a state from S_P combines with a state from S_Q to give a specific composite state.

These observations are particularly important when an automaton is to be realised as a direct product. Attention is then directed to homomorphic images of the given automaton, since each homomorphic image can be considered to represent a component semiautomaton in a direct product, and the realisation is based on mutually-resolving homomorphic images. Before considering composite realisations, however, the idea of an "ambiguity" should be regarded in a more general context. For example a group formalises a specific binary operation, and establishes the way two given group members g_1 and g_2 combine to define a member g_1g_2 [Fraleigh]. Then a quotient group can be regarded as a "simulation", since combining g_1 and g_2 can be simulated in the quotient group by combining the blocks B_1 and B_2 where $g_1 \in B_1$ and $g_2 \in B_2$, giving a block B_1B_2 where $g_1g_2 \in B_1B_2$. The block B_1B_2 is then an "ambiguity" regarding the solution obtained in the parent group, since B_1B_2 represents several possible solutions instead of the specific solution g_1g_2. The same idea arises in modular arithmetic, since a quotient ring Z/N is a homomorphic image of the parent ring Z. Then a calculation in the ring Z can be simulated by a calculation in Z/N, and this produces a solution p (mod N) representing an ambiguity $\{p, p+N, p+2N, \ldots\}$. The construction of Z/N as a direct product is considered in the "Chinese Remainder Theorem" [Stone], whereby Z/N is isomorphic to $Z/p_1 \times Z/p_2 \times Z/p_3 \times \ldots$, the numbers p_1, p_2, p_3 being the primes such that N is their product. In fact the direct-product construction is a standard topic from abstract algebra [Gratzer], for example a number of groups $G_1, G_2, G_3, \ldots$ can be combined to give a composite group $G_1 \times G_2 \times G_3 \times \ldots$, and then each component group is a homomorphic image. In contrast, the cascade construction seems of specific interest in computer science [Stone] and automata theory [Booth; Hartmanis & Stearns; Yoeli].

CHAPTER SIX: Composite Realisations of Finite Automata

6.1 Introduction

In considering automaton realisations using standard or "stock" sequential units, it was shown that a realisation can be based on an assignment $\langle \alpha \ \gamma \rangle$. Then γ is a one-many state assignment and α is a one-one input assignment, where γ is a one-many weak homomorphism under α.

A "composite" realisation of an automaton takes the form of interconnected sequential units, and a composite realisation can be formed if an assignment $\langle \alpha \ \gamma \rangle$ relates the objective semiautomaton to a composite semi-automaton. The initial aim is to consider the use of stock units to give realisations in the form of direct products. Then realisations in the form of cascade products will be considered, and "complex" realisations, involving the direct and cascade products together, will be introduced.

6.2 Product Realisations

In studying realisations in the form of direct products, called "product" realisations, attention is directed to images of the objective automaton. For example the objective automaton $\hat{J}$ from previously is reproduced as table (a) overleaf, and it is readily verified that $\pi_1 = (j_1\ j_2)(j_3)$ is a J-preserved S_J-cover. That is, π_1 is a cover of S_J and is preserved within the objective semiautomaton $J = \langle S_J \; \overline{X}_J \rangle$.

	x_1	x_2
j_1	$j_2/-$	$j_3/-$
j_2	$-/-$	j_3/z_1
j_3	j_2/z_3	j_3/z_2

<u>(a) Objective automaton</u>

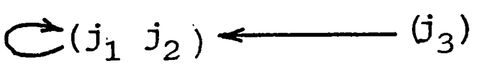
$\underline{\hat{J} = \langle S_J \; X_J \; Z_J \; \overline{X}_J \; \widetilde{X}_J \rangle}$

	x_1	x_2
p_1	p_2	p_3
p_2	p_1	p_3
p_3	p_2	p_3
p_4	p_1	p_4

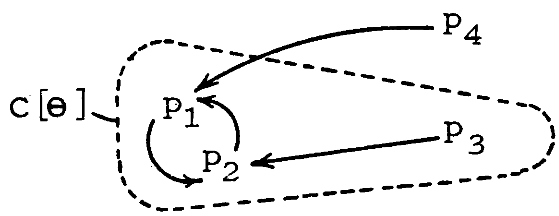
<u>(b) Stock semiautomaton</u>

$\underline{P = \langle S_P \; \overline{X}_P \rangle}$

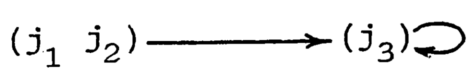
Furthermore a given preserved cover defines at least one image semiautomaton, and in particular $F_1 = \langle S_1 \; \overline{X}_1 \rangle$ is a π_1-image of semiautomaton J, where $S_1 = \pi_1$, $X_1 = X_J = \{x_1, x_2\}$ and the mappings $\overline{x_1}^1$, $\overline{x_2}^1$ over π_1 are those of figure 6.1 (a).



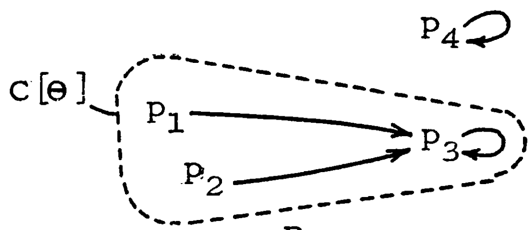
Mapping $\overline{x_1}^1$ over π_1



Mapping $\overline{x_1}^P$ over S_P



Mapping $\overline{x_2}^1$ over π_1



Mapping $\overline{x_2}^P$ over S_P

<u>(a)</u> $\underline{F_1 = \langle S_1 \; \overline{X}_1 \rangle}$

<u>(b)</u> $\underline{P = \langle S_P \; \overline{X}_P \rangle}$

<u>Figure 6.1</u>

In fact π_1 is a preserved "partition" rather than a preserved cover, and F_1 is the unique quotient semi-automaton associated with this partition, however this is unimportant and the essential feature is that F_1 is an image semiautomaton of J.

Assume now that the semiautomaton $P = \langle S_P \overline{X}_P \rangle$ of the above table (b) is a stock semiautomaton, so that P represents the state transitions of a sequential unit available from stock. Then the table expresses mappings $\overline{x}_1^P$ and $\overline{x}_2^P$ over S_P, as shown in figure 6.1(b), and the figure also shows that stock semiautomaton P is closely related to the image semiautomaton $F_1 = \langle S_1 \overline{X}_1 \rangle$. The relationship can be formalised as the relation θ from S_1 to S_P where

$$\theta = \{\langle (j_1\ j_2)\ p_1 \rangle\ \langle (j_1\ j_2)\ p_2 \rangle\ \langle (j_3)\ p_3 \rangle\},$$

and the one-many relation θ is expressed implicitly in figure 6.1 by arranging the codomain in accordance with θ. For example $p_1, p_2 \in S_P$ are adjacent since $\langle (j_1\ j_2)\ p_1 \rangle \in \theta$ and $\langle (j_1\ j_2)\ p_2 \rangle \in \theta$, and the figure also shows that the state $p_4 \in S_P$ is excluded from the codomain $C[\theta]$.

Then the relation θ is a one-many weak homomorphism of F_1 to P, that is θ has domain $D[\theta] = S_1$ and $(\forall x)(x \in X_J \Rightarrow \theta^{-1}\,\overline{x}^1 \subseteq \overline{x}^P\,\theta^{-1})$, as can be confirmed from figure 6.1. For example $\langle (j_1\ j_2)\ p_1 \rangle \in \theta$ so $\langle p_1\ (j_1\ j_2) \rangle \in \theta^{-1}$, and figure 6.1(a) shows

$\langle(j_1\ j_2)\ (j_1\ j_2)\rangle \in \overline{x_1}^1$ so $\langle p_1\ (j_1\ j_2)\rangle \in \theta^{-1}\overline{x_1}^1$. Then $\langle p_1\ (j_1\ j_2)\rangle \in \overline{x_1}^P\ \theta^{-1}$ is readily confirmed since $\langle p_1\ p_2\rangle \in \overline{x_1}^P$ and $\langle p_2\ (j_1\ j_2)\rangle \in \theta^{-1}$, and continuing this reasoning shows that θ is a one-many weak homomorphism. Consequently, each of the states p_1, p_2 and p_3 can be considered to represent an "ambiguity" regarding the objective automaton states. For example $\langle(j_1\ j_2)\ p_1\rangle \in \theta$, and this expresses that p_1 represents the subset $\{j_1, j_2\}$ of the objective state-set $S_J = \{j_1, j_2, j_3\}$

In effect θ assigns "ambiguities" to state-codes from S_P, instead of assigning the objective automaton states, and can be regarded as an "improper" state-assignment. Then P can be regarded as an improper realisation of the objective semiautomaton, and a product realisation is achieved by combining appropriate improper realisations, using the direct product. For example $\pi_6 = (j_1\ j_3)(j_2\ j_3)$ is a J-preserved S_J-cover and $F_6^- = \langle S_6^-\ \overline{X}_6\rangle$ is a π_6-image of semiautomaton J, where $S_6^- = \pi_6^-$, $X_6 = X_J = \{x_1, x_2\}$ and the mappings $\overline{x_1}^6$, $\overline{x_2}^6$ over π_6 are those of figure 6.2(a) Then the image semiautomaton F_6 is closely related to the semiautomaton $Q = \langle S_Q\ \overline{X}_Q\rangle$ of the table overleaf, and it is assumed that semiautomaton Q is a second stock semiautomaton, representing the state transitions of a sequential unit available from stock.

$(j_1\ j_3) \longrightarrow (j_2\ j_3)\circlearrowleft$

Mapping $\overline{x}_1^{\,6}$ over Π_6^-

$q_1 \longrightarrow q_2\circlearrowleft$

Mapping $\overline{x}_1^{\,Q}$ over S_Q

$\circlearrowright(j_1\ j_3) \qquad (j_2\ j_3)\circlearrowleft$

Mapping $\overline{x}_2^{\,6}$ over Π_6^-

$\circlearrowright q_1 \qquad q_2\circlearrowleft$

Mapping $\overline{x}_2^{\,Q}$ over S_Q

(a) Π_6^--image $F_6^- = \langle S_6^-\ \overline{X}_6^- \rangle$

(b) Stock semiautomaton $Q = \langle S_Q\ \overline{X}_Q \rangle$

Figure 6.2

	x_1	x_2
q_1	q_2	q_1
q_2	q_2	q_2

Stock semiautomaton

$Q = \langle S_Q\ \overline{X}_Q \rangle$

Here the relation $\zeta = \{\langle(j_1\ j_3)\ q_1\rangle\ \ \langle(j_2\ j_3)\ q_2\rangle\}$ is a one-many weak homomorphism of F_6^- to Q, and this can be verified by comparing the mappings $\overline{x}_1^{\,Q}$ and $\overline{x}_2^{\,Q}$ over S_Q with those over S_6^-, as in figure 6.2. Indeed ζ is an isomorphism, but this is unimportant and the real requirement is for ζ to be a one-many weak homomorphism.

Consequently each of the states $q_1, q_2 \in S_Q$ represents an ambiguity, and similarly for each of the states $p_1, p_2, p_3 \in S_P$, however the ambiguities are also "mutually resolving". For example figure 6.1 shows that p_1 represents the ambiguity $\{j_1\ j_2\}$, and q_1 represents the ambiguity $\{j_1\ j_3\}$ so the composite state

$\langle p_1\ q_1\rangle \in S_P \times S_Q$ represents j_1, this being the only objective state common to both ambiguities. Furthermore, ambiguities represented by respective states from S_P and S_Q will have at most one objective state in common, and this is expressed in the table.

		from S_P		
		p_1 $(j_1\ j_2)$	p_2 $(j_1\ j_2)$	p_3 (j_3)
from S_Q	q_1 $(j_1\ j_3)$	j_1	j_1	j_3
	q_2 $(j_2\ j_3)$	j_2	j_2	j_3

For example the co-ordinates $p_1 \in S_P$ and $q_1 \in S_Q$ give an entry j_1, since j_1 is the objective state common to the ambiguities represented by p_1 and q_1. Then the table expresses a natural relation γ from S_J to $S_P \times S_Q$ where

$$\gamma = \left\{ \begin{matrix} [j_1\ \ \langle p_1\ q_1\rangle] & [j_2\ \ \langle p_1\ q_2\rangle] & [j_3\ \ \langle p_3\ q_1\rangle] \\ [j_1\ \ \langle p_2\ q_1\rangle] & [j_2\ \ \langle p_2\ q_2\rangle] & [j_3\ \ \langle p_3\ q_2\rangle] \end{matrix} \right\}$$

so that γ relates an objective state to a composite state from $S_P \times S_Q$ if the components represent ambiguities with the objective state in common. For example the table shows that j_1 is common to the ambiguities represented by p_1 and q_1, so $[j_1\ \ \langle p_1\ q_1\rangle] \in \gamma$, and similarly $[j_1\ \ \langle p_2\ q_1\rangle] \in \gamma$.

Since γ is a relation from S_J to $S_P \times S_Q$, the relation establishes a correspondence between objective semiautomaton J and the direct product $P \times Q$. The direct product can be formalised as the semiautomaton $D = \langle S_D \; \overline{X}_D \rangle$ where $S_D = S_P \times S_Q$ and $X_D = X_J$, so that $x \in X_J$ implies $\overline{x}^D \in \overline{X}_D$ and mapping $\overline{x}^D$ over $S_P \times S_Q$ is defined as

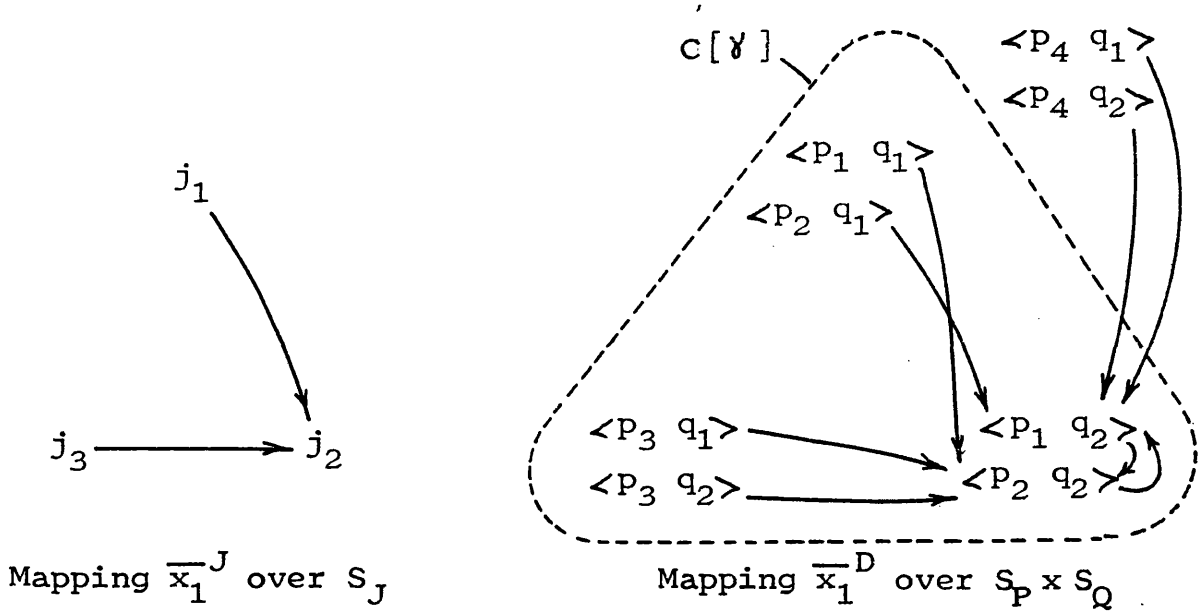
$$\overline{x}^D = \{[\langle p \; q\rangle \; \langle p' \; q'\rangle] \mid \langle p \; p'\rangle \in \overline{x}^P \;\;\&\;\; \langle q \; q'\rangle \in \overline{x}^Q\}.$$

Then $\overline{x_1}^D$, $\overline{x_2}^D$ are the mappings expressed in the table below, for example $[\langle p_1 \; q_1\rangle \;\; \langle p_2 \; q_2\rangle] \in \overline{x_1}^D$, since $\langle p_1 \; p_2\rangle \in \overline{x_1}^P$ from figure 6.1(b) and $\langle q_1 \; q_2\rangle \in \overline{x_1}^Q$ from figure 6.2(b).

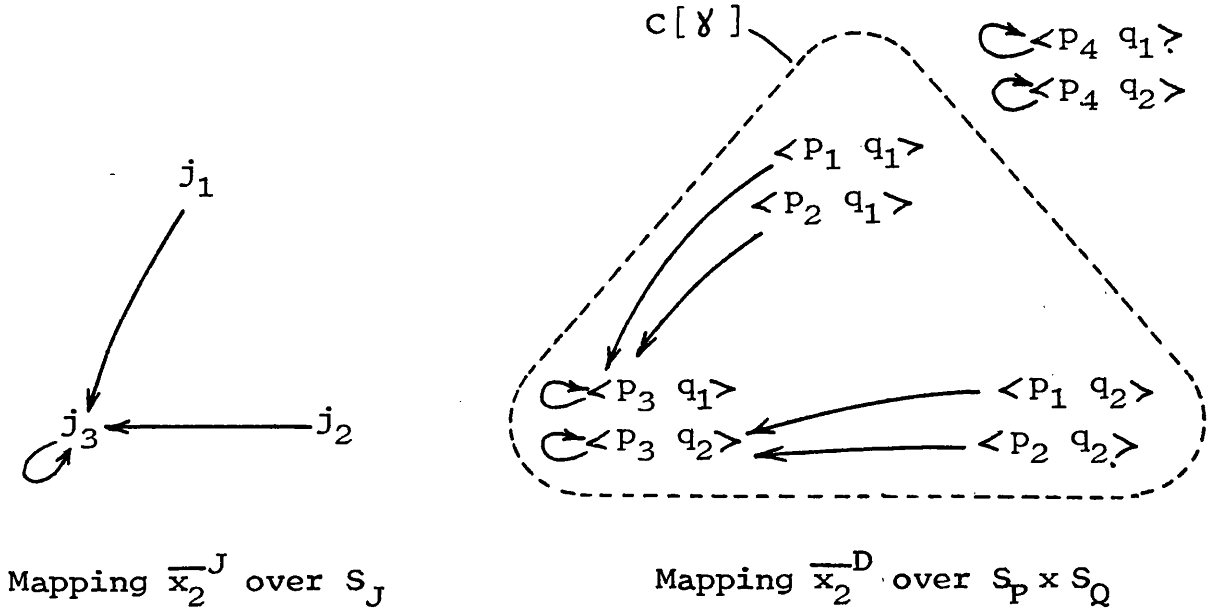
	x_1	x_2
$\langle p_1 \; q_1\rangle$	$\langle p_2 \; q_2\rangle$	$\langle p_3 \; q_1\rangle$
$\langle p_1 \; q_2\rangle$	$\langle p_2 \; q_2\rangle$	$\langle p_3 \; q_2\rangle$
$\langle p_2 \; q_1\rangle$	$\langle p_1 \; q_2\rangle$	$\langle p_3 \; q_1\rangle$
$\langle p_2 \; q_2\rangle$	$\langle p_1 \; q_2\rangle$	$\langle p_3 \; q_2\rangle$
$\langle p_3 \; q_1\rangle$	$\langle p_2 \; q_2\rangle$	$\langle p_3 \; q_1\rangle$
$\langle p_3 \; q_2\rangle$	$\langle p_2 \; q_2\rangle$	$\langle p_3 \; q_2\rangle$
$\langle p_4 \; q_1\rangle$	$\langle p_1 \; q_2\rangle$	$\langle p_4 \; q_1\rangle$
$\langle p_4 \; q_2\rangle$	$\langle p_1 \; q_2\rangle$	$\langle p_4 \; q_2\rangle$

<u>Direct Product $D = P \times Q$</u>

Then it is evident from figure 6.3 that γ is a one-many weak homomorphism of objective semiautomaton J to the direct product $P \times Q$.



(a) Showing $\gamma^{-1}\overline{x_1}^J \subseteq \overline{x_1}^D \cdot \gamma^{-1}$



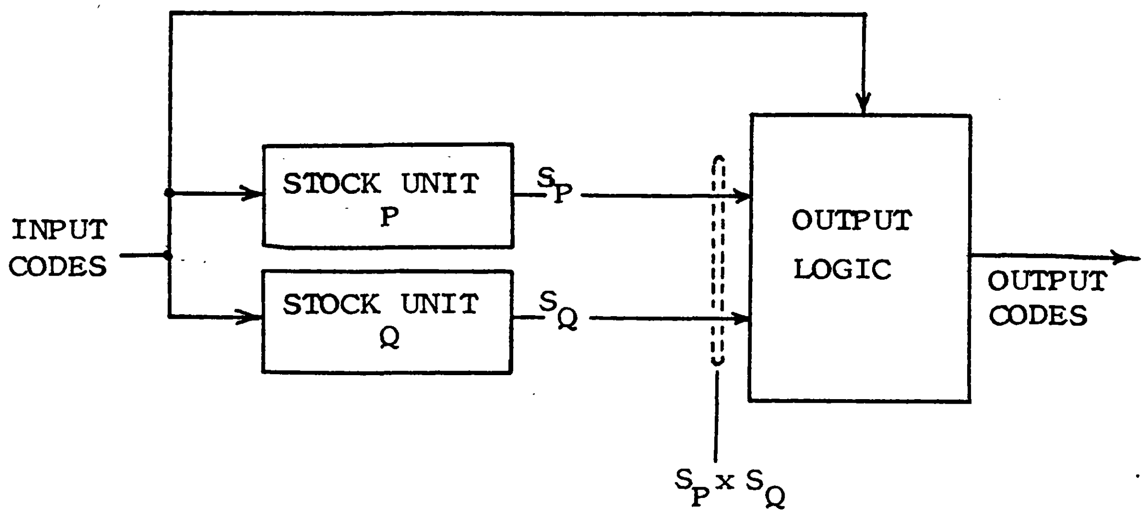
(b) Showing $\gamma^{-1}\overline{x_2}^J \subseteq \overline{x_2}^D\ \gamma^{-1}$

Figure 6.3

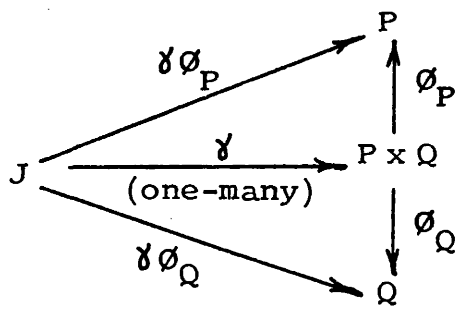
The one-many weak homomorphism of J to $P \times Q$, $J \leqslant^{\gamma} P \times Q$

Here the relation γ is expressed by arranging codomain $C[\gamma]$ accordingly, for example $\langle p_1\ q_1\rangle$ and $\langle p_2\ q_1\rangle$ lie together since $[j_1\ \ \langle p_1\ q_1\rangle] \in \gamma$ and $[j_1\ \ \langle p_2\ q_1\rangle] \in \gamma$. Then figure 6.3(a) shows $[\langle p_1\ q_1\rangle\ \ j_1] \in \gamma^{-1}$ and $\langle j_1\ j_2\rangle \in \overline{x_1}^J$, in which case $[\langle p_1\ q_1\rangle\ \ j_2] \in \gamma^{-1}\ \overline{x_1}^J$, furthermore $[\langle p_1\ q_1\rangle\ \ j_2] \in \overline{x_1}^D\ \gamma^{-1}$, in accordance with the inclusion $\gamma^{-1}\ \overline{x_1}^J \subseteq \overline{x_1}^D\ \gamma^{-1}$, since $[\langle p_1\ q_1\rangle\ \langle p_2\ q_2\rangle] \in \overline{x_1}^D$ and $[\langle p_2\ q_2\rangle\ \ j_2] \in \gamma^{-1}$. Similarly $\gamma^{-1}\ \overline{x_2}^J \subseteq \overline{x_2}^D\ \gamma^{-1}$, and $D[\gamma] = S_J$ so γ is a one-many weak homomorphism of J to $D = P \times Q$, that is $J \leqslant^{\gamma} P \times Q$.

This illustrates the way stock units can be used to form a product realisation, the realisation being based on stock semiautomata P and Q related to images of the objective semiautomaton. The preceeding has shown that a one-many weak homomorphism relates the image semiautomaton F_1 to the stock semiautomaton P, and image semiautomaton $F_{\bar{6}}$ is similarly related to the stock semiautomaton Q. Furthermore F_1 and $F_{\bar{6}}$ are "mutually resolving" weak-homorphic images of J, in the sense that the associated ambiguities have at most one objective state in common. Consequently a one-many weak homomorphism γ relates objective semiautomaton J to the direct product $D = P \times Q$, in which case the composite semiautomaton D can be used to form an automaton $\hat{D}$ so that $\hat{J} \leqslant \hat{D}$. Then the "product" realisation $\hat{D}$ will take the form of figure 6.4(a), and consists of the parallel interconnection of the units represented by the stock semiautomata P and Q, with associated combinatorial circuitry to produce the output codes. The corresponding weak-homomorphism diagram is shown in figure 6.4(b), where γ is the one-many weak homomorphism relating J to $P \times Q$, ϕ_P is the projection of $P \times Q$ onto P and similarly ϕ_Q is the projection of $P \times Q$ onto Q.



(a) Product Realisation $\hat{J} \leq \hat{D}$, where $D = P \times Q$



(b) Weak-homomorphism diagram

Figure 6.4

Then $J \xrightarrow{\gamma\phi_P} P$, showing that stock semiautomaton P is closely related to the objective semiautomaton, and similarly $J \xrightarrow{\gamma\phi_Q} Q$.

To develop this approach more formally, reconsider the J-preserved S_J-covers $\pi_1 = (j_1\ j_2)(j_3)$ and $\bar{\pi}_6 = (j_1\ j_3)(j_2\ j_3)$. These S_J-covers are mutually resolving since arbitrary blocks, one from each cover, have at most one objective state in common. That is either $B_1 \cap \bar{B}_6 = \emptyset$ or $B_1 \cap B_6$ is a singleton, for any cover blocks $B_1 \in \pi_1$ and $\bar{B}_6 \in \pi_6$, and this is evident from the "intersection" table.

		π_1: $(j_1\ j_2)$	π_1: (j_3)
π_6	$(j_1\ j_3)$	j_1	j_3
	$(j_2\ j_3)$	j_2	j_3

<u>Intersection table for S_J-covers π_1, π_6</u>

Here the entries represent the intersection of the co-ordinates, for example the j_1 entry expresses $\{j_1, j_2\} \cap \{j_1, j_3\} = \{j_1\}$. Clearly intersection always produces singletons, in the case of the covers π_1 and π_6, however mutually resolving covers will generally produce some void intersections and the null symbol $\emptyset$ will appear in the intersection table. To express the mutual-resolution property an operator $*$ can be introduced, as a way of combining arbitrary covers

π_i, π_j of a set S to give the cover $\pi_i * \pi_j$ where
$\pi_i * \pi_j = \{B \mid (\exists B_i)(\exists B_j)(B_i \in \pi_i, B_j \in \pi_j, B_i \cap B_j = B \ \& \ B \neq \emptyset)\}$.
Then mutual resolution of the S_J-covers π_1 and π_6 is expressed as $\pi_1 * \pi_6 = O(S_J)$, where $O(S_J) = (j_1)(j_2)(j_3)$ is the "zero cover" of $S_J = \{j_1, j_2, j_3\}$ and has a separate block for each objective state. More generally, a family $\{\pi_i\}_n$ of S_J-covers combine to give the S_J-cover $*\{\pi_i\}_n = \pi_0 * \pi_1 * \pi_2 * \ldots.$, and a family $\{A_i\}_n$ of image semiautomata $A_i = \langle S_i \ \overline{X}_i \rangle$ of $J = \langle S_J \ \overline{X}_J \rangle$ is a "mutually-resolving family" of image semiautomata if $*\{S_i\}_n = O(S_J)$.

The above intersection table expresses a natural relationship between S_J and the Cartesian product $\pi_1 \times \pi_6$, whereby an objective state $j \in S_J$ is related to a pair $<f_1 \ f_6> \in \pi_1 \times \pi_6$ if the blocks f_1 and f_6 intersect to give $\{j\}$. This can be formalised as a relation ν from S_J to $\pi_1 \times \pi_6$, where

$$\nu = \left\{ [j \ \ <f_1 \ f_6>] \ \middle| \ \begin{array}{c} j \in S_J, \ \ <f_1 \ f_6> \in \pi_1 \times \pi_6, \\ j \in f_1 \ \ \& \ \ j \in f_6 \end{array} \right\}$$

For example the intersection table shows that j_1 is common to block $(j_1 \ j_2)$ from π_1 and block $(j_1 \ j_3)$ from π_6, so $[j_1 \ \ <(j_1 \ j_2) \ (j_1 \ j_3)>] \in \nu$, and continuing this reasoning gives

$$\nu = \left\{ \begin{array}{ll} [j_1 \ \ <(j_1 \ j_2) \ (j_1 \ j_3)>] & [j_2 \ \ <(j_1 \ j_2) \ (j_2 \ j_3)>] \\ [j_3 \ \ <(j_3) \ (j_1 \ j_3)>] & [j_3 \ \ <(j_3) \ (j_2 \ j_3)>] \end{array} \right\}$$

This establishes the natural relationship between objective semiautomaton J and the direct product $F_1 \times F_6$ of the image semiautomata. Define $F = \langle S_F \; \overline{X}_F \rangle$ where $F = F_1 \times F_6$, so $S_F = \Pi_1 \times \Pi_6$, $X_F = X_J$, and $x \in X_J$ implies $\overline{x}^F \in \overline{X}_F$ where

$$\overline{x}^F = \left\{ [\langle f_1 \; f_6 \rangle \; \langle f_1' \; f_6' \rangle] \;\middle|\; \langle f_1 \; f_1' \rangle \; \in \overline{x}^1 \;\&\; \langle f_6 \; f_6' \rangle \; \in \overline{x}^6 \right\}$$

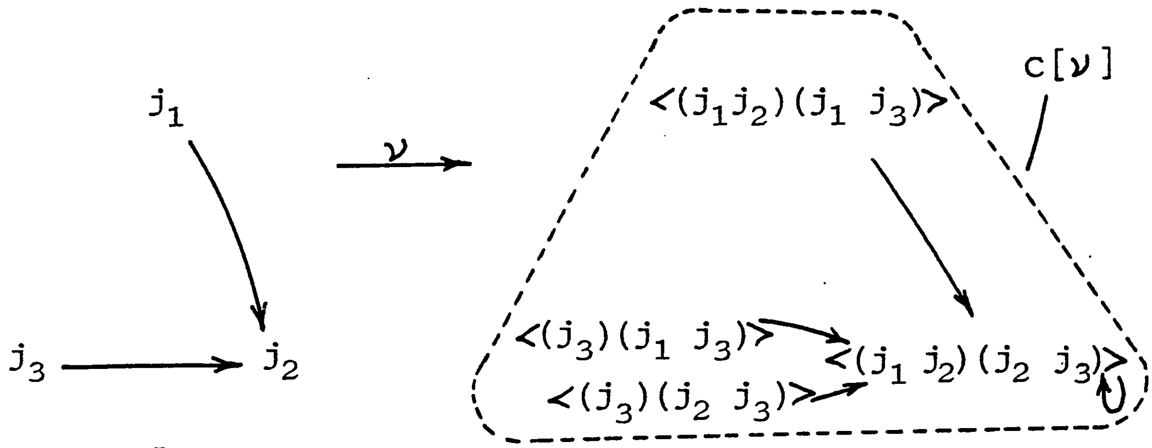
The resulting mappings $\overline{x_1}^F$ and $\overline{x_2}^F$ over $\Pi_1 \times \Pi_6$ are expressed in the table, in the normal way.

	x_1	x_2
$\langle(j_1\; j_2)\; (j_1\; j_3)\rangle$	$\langle(j_1\; j_2)\; (j_2\; j_3)\rangle$	$\langle(j_3)\; (j_1\; j_3)\rangle$
$\langle(j_1\; j_2)\; (j_2\; j_3)\rangle$	$\langle(j_1\; j_2)\; (j_2\; j_3)\rangle$	$\langle(j_3)\; (j_2\; j_3)\rangle$
$\langle(j_3)\; (j_1\; j_3)\rangle$	$\langle(j_1\; j_2)\; (j_2\; j_3)\rangle$	$\langle(j_3)\; (j_1\; j_3)\rangle$
$\langle(j_3)\; (j_2\; j_3)\rangle$	$\langle(j_1\; j_2)\; (j_2\; j_3)\rangle$	$\langle(j_3)\; (j_2\; j_3)\rangle$

<u>Direct Product $F = F_1 \times F_6$</u>

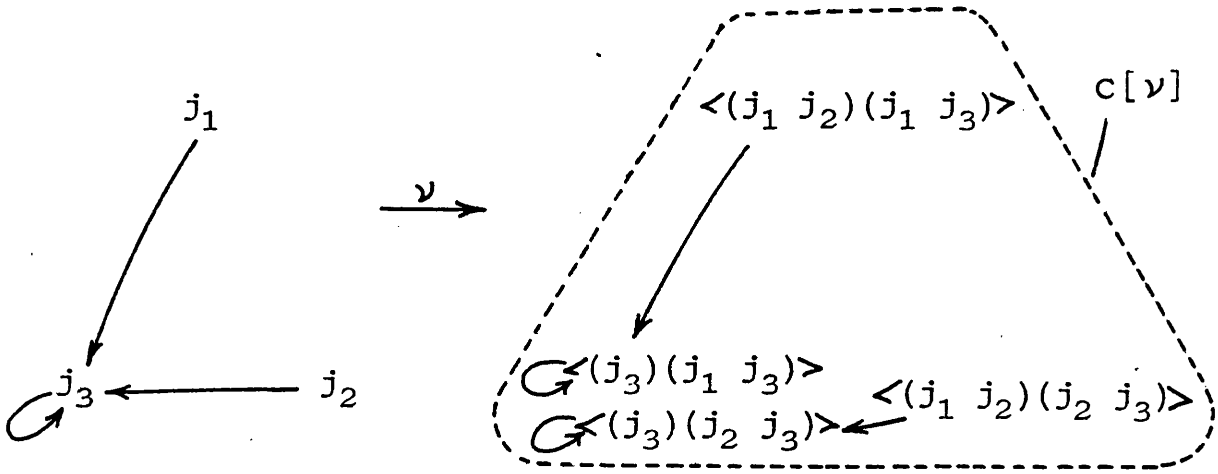
For example $[\langle(j_1\; j_2)\; (j_1\; j_3)\rangle \;\; \langle(j_1\; j_2)\; (j_2\; j_3)\rangle] \in \overline{x_1}^F$, since $\langle(j_1\; j_2)\; (j_1\; j_2)\rangle \in \overline{x_1}^1$ from figure 6.1 and $\langle(j_1\; j_3)\; (j_2\; j_3)\rangle \in \overline{x_1}^6$ from figure 6.2, and this is expressed by entering $\langle(j_1 \cdot j_2)\; (j_2\; j_3)\rangle$ in the x_1-column for the $\langle(j_1\; j_2)\; (j_1\; j_3)\rangle$ row.

Then the relation ν, establishing the correspondence between objective semiautomaton J and the product $F_1 \times F_6$, can be expressed by arranging the codomain $C[\nu]$ as in figure 6.5.



Mapping $\overline{x_1}^J$ over S_J Mapping $\overline{x_1}^F$ over $S_F = \pi_1 \times \pi_6$

(a) Showing $\nu^{-1}\overline{x_1}^J \subseteq \overline{x_1}^F \nu^{-1}$



Mapping $\overline{x_2}^J$ over S_J Mapping $\overline{x_2}^F$ over $S_F = \pi_1 \times \pi_6$

(b) Showing $\nu^{-1}\overline{x_2}^J \subseteq \overline{x_2}^F \nu^{-1}$

Figure 6.5

One-many weak homomorphism ν of J to $F = F_1 \times F_6$

For example figure 6.5(a) shows

$[j_1 \ \langle(j_1 \ j_2) \ (j_1 \ j_3)\rangle] \in \nu$, that is

$[\langle(j_1 \ j_2) \ (j_1 \ j_3)\rangle \ \ j_1] \in \nu^{-1}$, and shows $\langle j_1 \ j_2\rangle \in \overline{x_1}^J$

so $[\langle(j_1 \ j_2) \ (j_1 \ j_3)\rangle \ \ j_2] \in \nu^{-1} \ \overline{x_1}^J$. In addition the figure shows $[\langle(j_1 \ j_2) \ (j_1 \ j_3)\rangle \ \ \langle(j_1 \ j_2) \ (j_2 \ j_3)\rangle] \in \overline{x_1}^F$ and shows $[\langle(j_1 \ j_2) \ (j_2 \ j_3)\rangle \ \ j_2] \in \nu^{-1}$, so $[\langle(j_1 \ j_2) \ (j_1 \ j_3)\rangle \ \ j_2] \in \overline{x_1}^F \ \nu^{-1}$, and continuing this reasoning confirms $\nu^{-1} \ \overline{x_1}^J \subseteq \overline{x_1}^F \ \nu^{-1}$. Similarly figure 6.5(b) shows $\nu^{-1} \ \overline{x_2}^J \subseteq \overline{x_2}^F \ \nu^{-1}$, and ν is one-many with domain $D[\nu] = S_J$ so ν is a one-many weak homomorphism of J to $F_1 \times F_6$, that is $J \leqslant^{\nu} F_1 \times F_6$.

Theorem

Let $F_u = \langle S_u \ \overline{X}_u\rangle$ be a π_u-image of a semiautomaton $J = \langle S_J \ \overline{X}_J\rangle$, and let $F_v = \langle S_v \ \overline{X}_v\rangle$ be a π_v-image of J, where $\pi_u * \pi_v = 0(S_J)$. Then $J \leqslant F_u \times F_v$.

Proof

F_u is a π_u-image of J, so $S_u = \pi_u$ and $X_u = X_J$, similarly F_v is a π_v-image so $S_v = \pi_v$ and $X_v = X_J$. Define $D = \langle S_D \ \overline{X}_D\rangle$ where $D = F_u \times F_v$, so $S_D = S_u \times S_v = \pi_u \times \pi_v$, $X_D = X_J$, and $x \in X_J$ implies $\overline{x}^D \in \overline{X}_D$ where

$$\overline{x}^D = \left\{[\langle f_u \ f_v\rangle \ \ \langle f'_u \ f'_v\rangle] \;\middle|\; \langle f_u \ f'_u\rangle \in \overline{x}^u \ \& \ \langle f_v \ f'_v\rangle \in \overline{x}^v\right\}.$$

Define relation ν from S_J to $\pi_u \times \pi_v$ where, for Π'_u the canonical relation from S_J to π_u and Π'_v the canonical relation from S_J to π_v,

$$\nu = \left\{[j \ \ \langle f_u \ f_v\rangle] \;\middle|\; \langle j \ f_u\rangle \in \Pi'_u \ \& \ \langle j \ f_v\rangle \in \Pi'_v\right\}.$$

Assume $x \in X_J$, and assume $[\langle f_u\ f_v\rangle\ j'] \in \nu^{-1}\bar{x}^J$, so $[\langle f_u\ f_v\rangle\ j] \in \nu^{-1}$ and $\langle j\ j'\rangle \in \bar{x}^J$ for some j. Then $[j\ \langle f_u\ f_v\rangle] \in \nu$, so $\langle j\ f_u\rangle \in \Pi'_u$ and $\langle j\ f_v\rangle \in \Pi'_v$. Consequently $\langle f_u\ j\rangle \in \Pi'^{-1}_u$, and $\langle j\ j'\rangle \in \bar{x}^J$ so $\langle f_u\ j'\rangle \in \Pi'^{-1}_u\bar{x}^J$. Since F_u is a π_u-image the canonical relation Π'_u is a weak homomorphism of J to F_u, hence $\Pi'^{-1}_u\bar{x}^J \subseteq \bar{x}^u\Pi'^{-1}_u$ so $\langle f_u\ j'\rangle \in \Pi'^{-1}_u\bar{x}^J$ implies $\langle f_u\ j'\rangle \in \bar{x}^u\Pi'^{-1}_u$, in which case $\langle f_u\ f'_u\rangle \in \bar{x}^u$ and $\langle f'_u\ j'\rangle \in \Pi'^{-1}_u$, that is $\langle j'\ f'_u\rangle \in \Pi'_u$, for some f'_u. Similarly $\langle j\ f_v\rangle \in \Pi'_v$ implies $\langle f_v\ j\rangle \in \Pi'^{-1}_v$, and $\langle j\ j'\rangle \in \bar{x}^J$ so $\langle f_v\ j'\rangle \in \Pi'^{-1}_v\bar{x}^J$, furthermore Π'_v is a weak homomorphism of J to π_v-image F_v so $\Pi'^{-1}_v\bar{x}^J \subseteq \bar{x}^v\Pi'^{-1}_v$. Consequently $\langle f_v\ j'\rangle \in \bar{x}^v\Pi'^{-1}_v$, in which case $\langle f_v\ f'_v\rangle \in \bar{x}^v$ and $\langle f'_v\ j'\rangle \in \Pi'^{-1}_v$, that is $\langle j'\ f'_v\rangle \in \Pi'_v$, for some f'_v.

Hence $\langle f_u\ f'_u\rangle \in \bar{x}^u$ and $\langle f_v\ f'_v\rangle \in \bar{x}^v$, in which case $[\langle f_u\ f_v\rangle\ \langle f'_u\ f'_v\rangle] \in \bar{x}^D$, furthermore $\langle j'\ f'_u\rangle \in \Pi'_u$ and $\langle j'\ f'_v\rangle \in \Pi'_v$ so $[j'\ \langle f'_u\ f'_v\rangle] \in \nu$. Equivalently $[\langle f'_u\ f'_v\rangle\ j'] \in \nu^{-1}$, and $[\langle f_u\ f_v\rangle\ \langle f'_u\ f'_v\rangle] \in \bar{x}^D$ so $[\langle f_u\ f_v\rangle\ j'] \in \bar{x}^D\nu^{-1}$. Therefore $[\langle f_u\ f_v\rangle\ j'] \in \nu^{-1}\bar{x}^J$ implies $[\langle f_u\ f_v\rangle\ j'] \in \bar{x}^D\nu^{-1}$, and $x \in X_J$ is arbitrary so $(\forall x)(x \in X_J \Longrightarrow \nu^{-1}\bar{x}^J \subseteq \bar{x}^D\nu^{-1})$.

Considering now the nature of relation ν clearly $D[\nu] \subseteq S_J$, furthermore $j \in S_J$ implies $\langle j\ f_u\rangle \in \Pi'_u$ for some f_u since π_u is a S_J-cover, similarly $\langle j\ f_v\rangle \in \pi_v$ for some f_v. Then $[j\ \langle f_u\ f_v\rangle] \in \nu$, so $j \in D[\nu]$, and this confirms $S_J \subseteq D[\nu]$. Therefore $D[\nu] = S_J$, and from above $\nu^{-1}\bar{x}^J \subseteq \bar{x}^D\nu^{-1}$ for any $x \in X_J$ so ν is a weak homomorphism of J to $F_u \times F_v$. To confirm ν to be one-many, assume $[\langle f_u\ f_v\rangle\ j] \in \nu^{-1}$ and assume $[\langle f_u\ f_v\rangle\ j^*] \in \nu^{-1}$. Then $[j\ \langle f_u\ f_v\rangle] \in \nu$, so $\langle j\ f_u\rangle \in \pi_u$ and $\langle j\ f_v\rangle \in \pi_v$, that is $j \in f_u$ and $j \in f_v$. Consequently $j \in f_u \cap f_v$, indeed $f_u \cap f_v = \{j\}$ since $\pi_u * \pi_v = O(S_J)$. Similarly $[j^*\ \langle f_u\ f_v\rangle] \in \nu$ so $j^* \in f_u$ and $j^* \in f_v$, that is $j^* \in f_u \cap f_v$ so $\{j^*\} = f_u \cap f_v = \{j\}$, in which case $j = j^*$. Hence $[\langle f_u\ f_v\rangle\ j] \in \nu^{-1}$ and $[\langle f_u\ f_v\rangle\ j^*] \in \nu^{-1}$ implies $j = j^*$, so ν^{-1} is a mapping. Consequently ν is one-many, and ν is a weak homomorphism of J to $F_u \times F_v$ so $J \leqslant^{\nu} F_u \times F_v$, completing the proof.

The theorem formalises the idea of combining mutually-resolving ambiguous representations, to give a precise simulation. In the example the image semiautomata F_1 and F_6 are ambiguous representations of the semiautomaton J, in the same way that a quotient group is an ambiguous representation of the parent group. However $\pi_1 * \pi_6 = O(S_J)$ ensures that the ambiguous representations are "mutually resolving", so the image semiautomata F_1 and F_6 combine to give a precise simulation $J \leqslant F_1 \times F_6$, as shown in figure 6.5. Furthermore, this idea can be extended to any given family of mutually-resolving image semiautomata. For simplicity the theorem treats the restricted case of combining just two image semiautomata, but in general a given semiautomaton can be covered by any number of mutually-resolving images. For example if π_u, π_v, π_w are J-preserved S_J-covers where $\pi_u * \pi_v * \pi_w = 0(S_J)$, and F_u, F_v, F_w are corresponding image semiautomata, then $J \leqslant F_u \times F_v \times F_w$.

The covering $J \leqslant F_1 \times F_6$ is an important step in forming a realisation of objective automaton $\hat{J}$, since $F_1 \times F_6$ is closely related to the direct product $P \times Q$ of the stock semiautomata. From previously the relation θ from π_1 to S_P is a one-many weak homomorphism of image semiautomaton F_1 to the stock semiautomaton P, so to any ambiguity $f_1 \in \pi_1$ corresponds at least one state-code $p \in S_P$ where $\langle f_1\ p \rangle \in \theta$. Similarly the relation ζ from π_6 to S_Q is a one-many weak homomorphism of F_6 to stock semiautomaton Q, so to any ambiguity $f_6 \in \pi_6$ corresponds some state-code $q \in S_Q$ where $\langle f_6\ q \rangle \in \zeta$. Consequently, to the pair $\langle f_1\ f_6 \rangle \in \pi_1 \times \pi_6$ corresponds at least one pair $\langle p\ q \rangle \in S_P \times S_Q$ where $\langle f_1\ p \rangle \in \theta$ and $\langle f_6\ q \rangle \in \zeta$. This can be formalised as a relation μ from $\pi_1 \times \pi_6$ to $S_P \times S_Q$ given by

$$\mu = \left\{ [\ \langle f_1\ f_6 \rangle\ \ \langle p\ q \rangle\ ]\ \middle|\ \langle f_1\ p \rangle \in \theta\ \ \&\ \ \langle f_6\ q \rangle \in \zeta \right\},$$

where from previously

$$\theta = \left\{ \langle (j_1\ j_2)\ p_1 \rangle\ \langle (j_1\ j_2)\ p_2 \rangle\ \langle (j_3)\ p_3 \rangle \right\}$$
$$\zeta = \left\{ \langle (j_1\ j_3)\ q_1 \rangle\ \langle (j_2\ j_3)\ q_2 \rangle \right\}$$

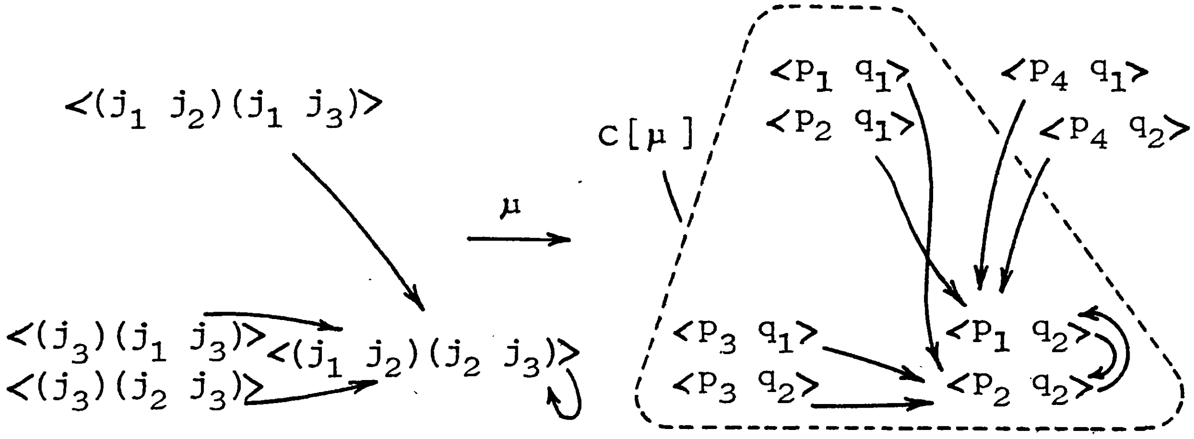
so μ =

$$\left\{\begin{array}{ll} [\langle (j_1\ j_2)\ (j_1\ j_3)\rangle\ \ \langle p_1\ q_1\rangle] & [\langle (j_1\ j_2)\ (j_2\ j_3)\rangle\ \ \langle p_1\ q_2\rangle] \\ [\langle (j_1\ j_2)\ (j_1\ j_3)\rangle\ \ \langle p_2\ q_1\rangle] & [\langle (j_1\ j_2)\ (j_2\ j_3)\rangle\ \ \langle p_2\ q_2\rangle] \\ [\langle (j_3)\ (j_1\ j_3)\rangle\ \ \langle p_3\ q_1\rangle] & [\langle (j_3)\ (j_2\ j_3)\rangle\ \ \langle p_3\ q_2\rangle] \end{array}\right\}.$$

Clearly μ is a one-many relation from $\pi_1 \times \pi_6$ to $S_P \times S_Q$, with domain $D[\mu] = \pi_1 \times \pi_6$. Furthermore μ relates the direct product $F_1 \times F_6$ to the direct product $P \times Q$ of the stock semiautomata, and this is illustrated in figure 6.6. For example figure 6.6(a) shows that μ relates

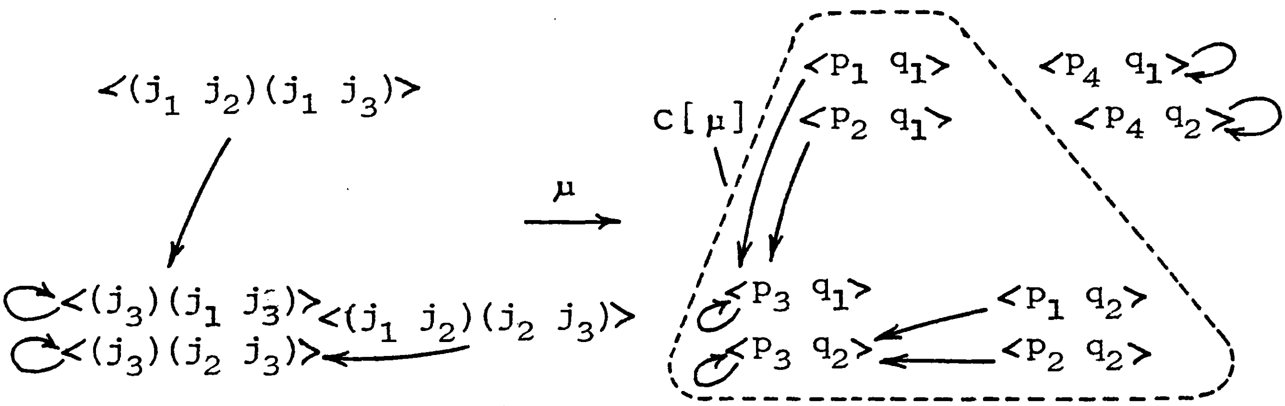
$\langle (j_1\ j_2)\ (j_1\ j_3)\rangle \in \pi_1 \times \pi_6$ to $\langle p_1\ q_1\rangle \in S_P \times S_Q$, so $[\langle p_1\ q_1\rangle\ \ \langle (j_1\ j_2)\ (j_1\ j_3)\rangle\ ] \in \mu^{-1}$, and shows $[\ \langle (j_1\ j_2)\ (j_1\ j_3)\rangle\ \langle (j_1\ j_2)\ (j_2\ j_3)\rangle\ ] \in \overline{x_1}^F$ so $[\ \langle p_1\ q_1\rangle\ \ \langle (j_1\ j_2)\ (j_2\ j_3)\rangle\ ] \in \mu^{-1}\,\overline{x_1}^F$. In addition figure 6.6(a) shows $[\langle p_1\ q_1\rangle\ \ \langle p_2\ q_2\rangle\ ] \in \overline{x_1}^D$, and shows $[\langle p_2\ q_2\rangle\ \ \langle (j_1\ j_2)\ (j_2\ j_3)\rangle\ ] \in \mu^{-1}$ so $[\ \langle p_1\ q_1\rangle\ \ \langle (j_1\ j_2)\ (j_2\ j_3)\rangle\ ] \in \overline{x_1}^D\,\mu^{-1}$, in accordance with the inclusion $\mu^{-1}\,\overline{x_1}^F \subseteq \overline{x_1}^D\,\mu^{-1}$. Continuing this reasoning confirms $\mu^{-1}\,\overline{x_1}^F \subseteq \overline{x_1}^D\,\mu^{-1}$, and similarly figure 6.6(b) shows $\mu^{-1}\,\overline{x_2}^F \subseteq \overline{x_2}^D\,\mu^{-1}$.

Consequently μ is a one-many weak homomorphism of $F = F_1 \times F_6$ to $P \times Q$, that is $F_1 \times F_6 \leqslant^{\mu} P \times Q$. More generally, if F_u, F_v, P and Q are any given semiautomata where F_u is covered by P and F_v is covered by Q, then $F_u \times F_v$ is covered by $P \times Q$.



Mapping $\overline{x_1}^F$ over $\Pi_1 \times \Pi_6$ Mapping $\overline{x_1}^D$ over $S_P \times S_Q$

(a) $\mu^{-1}\overline{x_1}^F \subseteq \overline{x_1}^D \mu^{-1}$



Mapping $\overline{x_2}^F$ over $\Pi_1 \times \Pi_6$ Mapping $\overline{x_2}^D$ over $S_P \times S_Q$

(b) $\mu^{-1}\overline{x_2}^F \subseteq \overline{x_2}^D \mu^{-1}$

Figure 6.6 $F_1 \times F_6 \leq^{\mu} P \times Q$

Theorem

If $F_u \leqslant P$ and $F_v \leqslant Q$ then $F_u \times F_v \leqslant P \times Q$

Proof

Define $F = \langle S_F\ \overline{X}_F \rangle$ where $F = F_u \times F_v$, and define $D = \langle S_D\ \overline{X}_D \rangle$ where $D = P \times Q$. Assuming $F_u \leqslant^{\theta} P$ and $F_v \leqslant^{\varsigma} Q$, where $F_u = \langle S_u\ \overline{X}_u \rangle$, $F_v = \langle S_v\ \overline{X}_v \rangle$, $P = \langle S_P\ \overline{X}_P \rangle$ and $Q = \langle S_Q\ \overline{X}_Q \rangle$ define the relation μ from $S_u \times S_v$ to $S_P \times S_Q$ where

$$\mu = \left\{ [\langle f_u\ f_v \rangle \quad \langle p\ q \rangle] \,\middle|\, \langle f_u\ p \rangle \in \theta \text{ and } \langle f_v\ q \rangle \in \varsigma \right\}$$

and define $X = X_u = X_v = X_P = X_Q = X_F = X_D$.

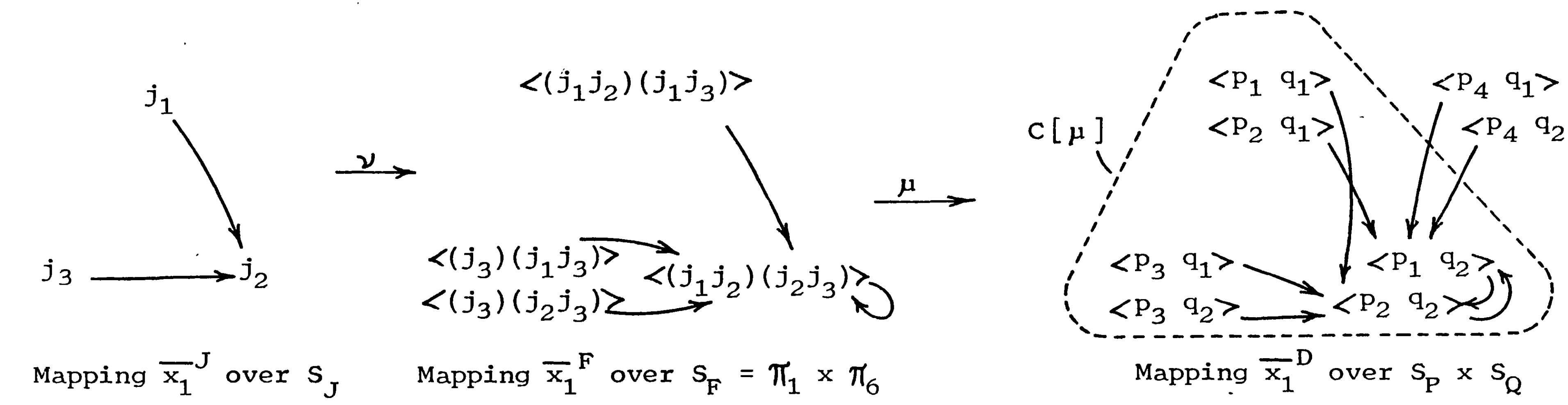
Assume $x \in X$ and assume $[\langle p\ q \rangle \quad \langle f_u'\ f_v' \rangle] \in \mu^{-1} \overline{x}^F$, so $[\langle p\ q \rangle \quad \langle f_u\ f_v \rangle] \in \mu^{-1}$ and $[\langle f_u\ f_v \rangle \quad \langle f_u'\ f_v' \rangle] \in \overline{x}^F$ for some $\langle f_u\ f_v \rangle$. Then $[\langle f_u\ f_v \rangle \quad \langle p\ q \rangle] \in \mu$, so $\langle f_u\ p \rangle \in \theta$ and $\langle f_v\ q \rangle \in \varsigma$, furthermore $[\langle f_u\ f_v \rangle \quad \langle f_u'\ f_v' \rangle] \in \overline{x}^F$ where $F = F_u \times F_v$, so $\langle f_u\ f_u' \rangle \in \overline{x}^u$ and $\langle f_v\ f_v' \rangle \in \overline{x}^v$. Therefore $\langle p\ f_u \rangle \in \theta^{-1}$ and $\langle f_u\ f_u' \rangle \in \overline{x}^u$, in which case $\langle p\ f_u' \rangle \in \theta^{-1}\, \overline{x}^u$, and $F_u \leqslant^{\theta} P$ so $\theta^{-1} \overline{x}^u \subseteq \overline{x}^P \theta^{-1}$. Consequently $\langle p\ f_u' \rangle \in \overline{x}^P \theta^{-1}$, so $\langle p\ p' \rangle \in \overline{x}^P$ and $\langle p'\ f_u' \rangle \in \theta^{-1}$ for some p'. Similarly $\langle q\ f_v \rangle \in \varsigma^{-1}$ and $\langle f_v\ f_v' \rangle \in \overline{x}^v$ gives $\langle q\ f_v' \rangle \in \varsigma^{-1} \overline{x}^v$, and $F_v \leqslant^{\varsigma} Q$ so $\varsigma^{-1} \overline{x}^v \subseteq \overline{x}^Q \varsigma^{-1}$. Therefore $\langle q\ f_v' \rangle \in \overline{x}^Q \varsigma^{-1}$, so $\langle q\ q' \rangle \in \overline{x}^Q$ and $\langle q'\ f_v' \rangle \in \varsigma^{-1}$ for some q'.

By definition $D = P \times Q$, and from above $\langle p\ p' \rangle \in \overline{x}^P$ and $\langle q\ q' \rangle \in \overline{x}^Q$ so

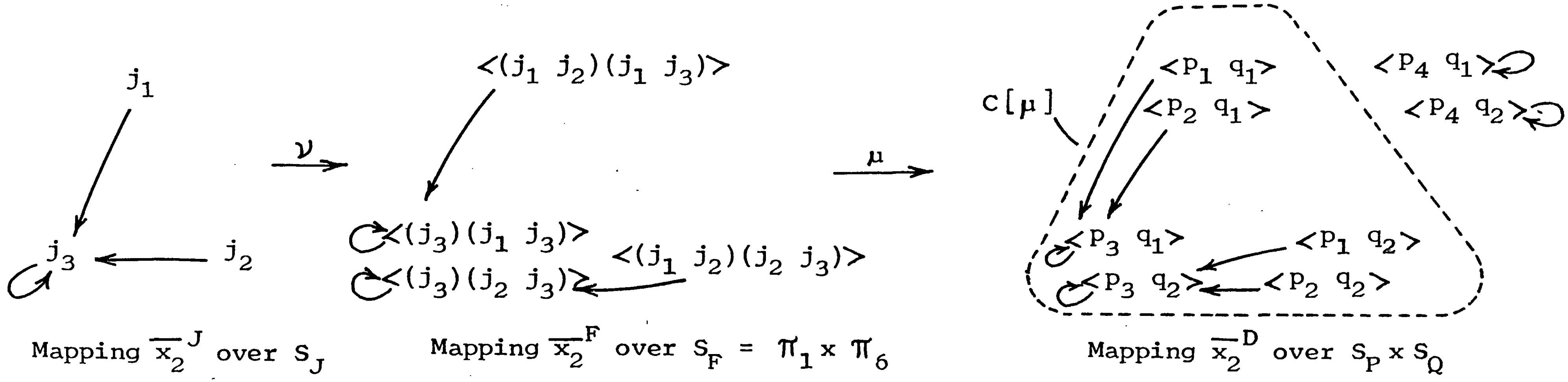
$[\langle p\ q\rangle\ \ \langle p'\ q'\rangle] \in \overline{x}^D$. Furthermore $\langle f'_u\ p'\rangle \in \theta$ and $\langle f'_v\ q'\rangle \in \mathfrak{S}$ so $[\langle f'_u\ f'_v\rangle\ \ \langle p'\ q'\rangle] \in \mu$, that is $[\langle p'\ q'\rangle\ \ \langle f'_u\ f'_v\rangle] \in \mu^{-1}$, therefore $[\langle p\ q\rangle\ \ \langle f'_u\ f'_v\rangle] \in \overline{x}^D \mu^{-1}$. Hence $[\langle p\ q\rangle\ \ \langle f'_u\ f'_v\rangle] \in \mu^{-1}\overline{x}^F$ implies $[\langle p\ q\rangle\ \ \langle f'_u\ f'_v\rangle] \in \overline{x}^D \mu^{-1}$, and $x \in X$ is arbitrary so $(\forall x)\ (x \in X \Longrightarrow \mu^{-1}\overline{x}^F \subseteq \overline{x}^D \mu^{-1})$.

Considering now the nature of μ clearly $D[\mu] \subseteq S_u \times S_v$, furthermore $\langle f_u\ f_v\rangle \in S_u \times S_v$ implies $f_u \in S_u$ and $f_v \in S_v$, where $S_u = D[\theta]$ and $S_v = D[\mathfrak{S}]$. Consequently $\langle f_u\ p\rangle \in \theta$ for some p, and $\langle f_v\ q\rangle \in \mathfrak{S}$ for some q, in which case $[\langle f_u\ f_v\rangle\ \ \langle p\ q\rangle] \in \mu$. Hence $\langle f_u\ f_v\rangle \in S_u \times S_v$ implies $\langle f_u\ f_v\rangle \in D[\mu]$, giving $S_u \times S_v \subseteq D[\mu]$, and from above $D[\mu] \subseteq S_u \times S_v$ so $D[\mu] = S_u \times S_v$. Furthermore $\mu^{-1}\overline{x}^F \subseteq \overline{x}^D \mu^{-1}$ for any $x \in X_F$, so μ is confirmed to be a weak homomorphism of $F_u \times F_v$ to $P \times Q$. To prove μ to be one-many, assume $[\langle p\ q\rangle\ \ \langle f_u\ f_v\rangle] \in \mu^{-1}$ and assume $[\langle p\ q\rangle\ \ \langle f_u^*\ f_v^*\rangle] \in \mu^{-1}$. Then $[\langle f_u\ f_v\rangle\ \ \langle p\ q\rangle] \in \mu$, so $\langle f_u\ p\rangle \in \theta$ and $\langle f_v\ q\rangle \in \mathfrak{S}$, furthermore $[\langle f_u^*\ f_v^*\rangle\ \ \langle p\ q\rangle] \in \mu$ so $\langle f_u^*\ p\rangle \in \theta$ and $\langle f_v^*\ q\rangle \in \mathfrak{S}$. Therefore $\langle p\ f_u\rangle, \langle p\ f_u^*\rangle \in \theta^{-1}$, however $F_u \leqslant^\theta P$ so θ^{-1} is a mapping, and then $\langle p\ f_u\rangle, \langle p\ f_u^*\rangle \in \theta^{-1}$ implies $f_u = f_u^*$. Similarly $\langle q\ f_v\rangle, \langle q\ f_v^*\rangle \in \mathfrak{S}^{-1}$ where $\mathfrak{S}^{-1}$ is a mapping so $f_v = f_v^*$, giving $\langle f_u\ f_v\rangle = \langle f_u^*\ f_v^*\rangle$. Hence $[\langle p\ q\rangle\ \ \langle f_u\ f_v\rangle] \in \mu^{-1}$ and $[\langle p\ q\rangle\ \ \langle f_u^*\ f_v^*\rangle] \in \mu^{-1}$ implies $\langle f_u\ f_v\rangle = \langle f_u^*\ f_v^*\rangle$, and this confirms that μ^{-1} is a mapping. Consequently μ is one-many, so μ is a one-many weak homomorphism of $F_u \times F_v$ to $P \times Q$, that is $F_u \times F_v \leqslant^{\mu} P \times Q$, completing the proof.

From previously the image semiautomaton F_1 is covered under θ by the stock semiautomaton P, that is $F_1 \leqslant^{\theta} P$, and similarly $F_6 \leqslant^{\xi} Q$, so by the theorem $F_1 \times F_6 \leqslant P \times Q$. This would be the case for any given semiautomata F_1 and F_6 where $F_1 \leqslant^{\theta} P$ and $F_6 \leqslant^{\xi} Q$, since the semiautomata F_u and F_v of the theorem are not necessarily image semiautomata. Here however F_1 and F_6 are image semiautomata and they are also mutually-resolving, so by the previous theorem $J \leqslant F_1 \times F_6$. Consequently $J \leqslant^{\nu} F_1 \times F_6 \leqslant^{\mu} P \times Q$, and covering is transitive so $J \leqslant^{\nu\mu} P \times Q$ as shown in figure 6.7, where $P \times Q = D = \langle S_D \; \overline{X_D} \rangle$ as in figure 6.3. In effect $P \times Q$ "realises" J indirectly, by realising the product $F_1 \times F_6$, where $F_1 \times F_6$ covers semiautomaton J since F_1 and F_6 are mutually-resolving images. For example figure 6.7(a) shows that the association $[\langle p_1 \; q_1 \rangle \;\; \langle p_2 \; q_2 \rangle] \in \overline{x_1}^D$ represents the association $[\langle (j_1 \; j_2) \; (j_1 \; j_3) \rangle \;\; \langle (j_1 \; j_2) \; (j_2 \; j_3) \rangle] \in \overline{x_1}^F$, which in turn represents the association $\langle j_1 \; j_2 \rangle \in \overline{x_1}^J$, so the composite state-code changes from $\langle p_1 \; q_1 \rangle$ to $\langle p_2 \; q_2 \rangle$ in simulation of the objective transition $\langle j_1 \; j_2 \rangle \in \overline{x_1}^J$. The figures 6.3 and 6.7 are directly related since $\gamma = \nu \mu$, and figure 6.7 can be considered



(a) $\nu^{-1}\overline{x}_1^{\,J} \subseteq \overline{x}_1^{\,F}\nu^{-1}$ and $\mu^{-1}\overline{x}_1^{\,F} \subseteq \overline{x}_1^{\,D}\mu^{-1}$



(b) $\nu^{-1}\overline{x}_2^{\,J} \subseteq \overline{x}_2^{\,F}\nu^{-1}$ and $\mu^{-1}\overline{x}_2^{\,F} \subseteq \overline{x}_2^{\,D}\mu^{-1}$

Figure 6.7 $J \leq^{\nu\mu} P \times Q$

to express the detail omitted from figure 6.3.

The property $J \leqslant^{\nu\mu} P \times Q$ ensures that the parallel interconnection of the stock units, together with circuitry to produce output codes, gives a product realisation of objective automaton $\hat{J}$, and this has been shown in figure 6.4(a). The corresponding weak homomorphism graph is that of figure 6.8, and this is closely related to figure 6.4(b) since $\gamma = \nu\mu$. Additional weak homomorphisms can be formed on figure 6.8 by taking products, for example the relationship between the objective semiautomaton J and the component semiautomaton P can be expressed as $J \xrightarrow{\nu\phi_1\theta} P$ or as $J \xrightarrow{\nu\mu\phi_P} P$, where ϕ_1 is the projection of $F_1 \times F_6$ onto F_1 and ϕ_P is the projection of $P \times Q$ onto P.

$$
\begin{tikzcd}
 & F_1 \arrow[rr, "\theta"] & & P \\
J \arrow[r, "\nu"] \arrow[rrr, bend right, "\gamma"'] & F_1 \times F_6 \arrow[u, "\phi_1"] \arrow[d, "\phi_6"] \arrow[rr, "\mu"] & & P \times Q \arrow[u, "\phi_P"'] \arrow[d, "\phi_Q"'] \\
 & F_6 \arrow[rr, "\xi"] & & Q
\end{tikzcd}
$$

Weak-homomorphism graph for $J \leqslant^{\nu} F_1 \times F_6 \leqslant^{\mu} P \times Q$ Figure 6.8

6.3 Stock-unit assessment

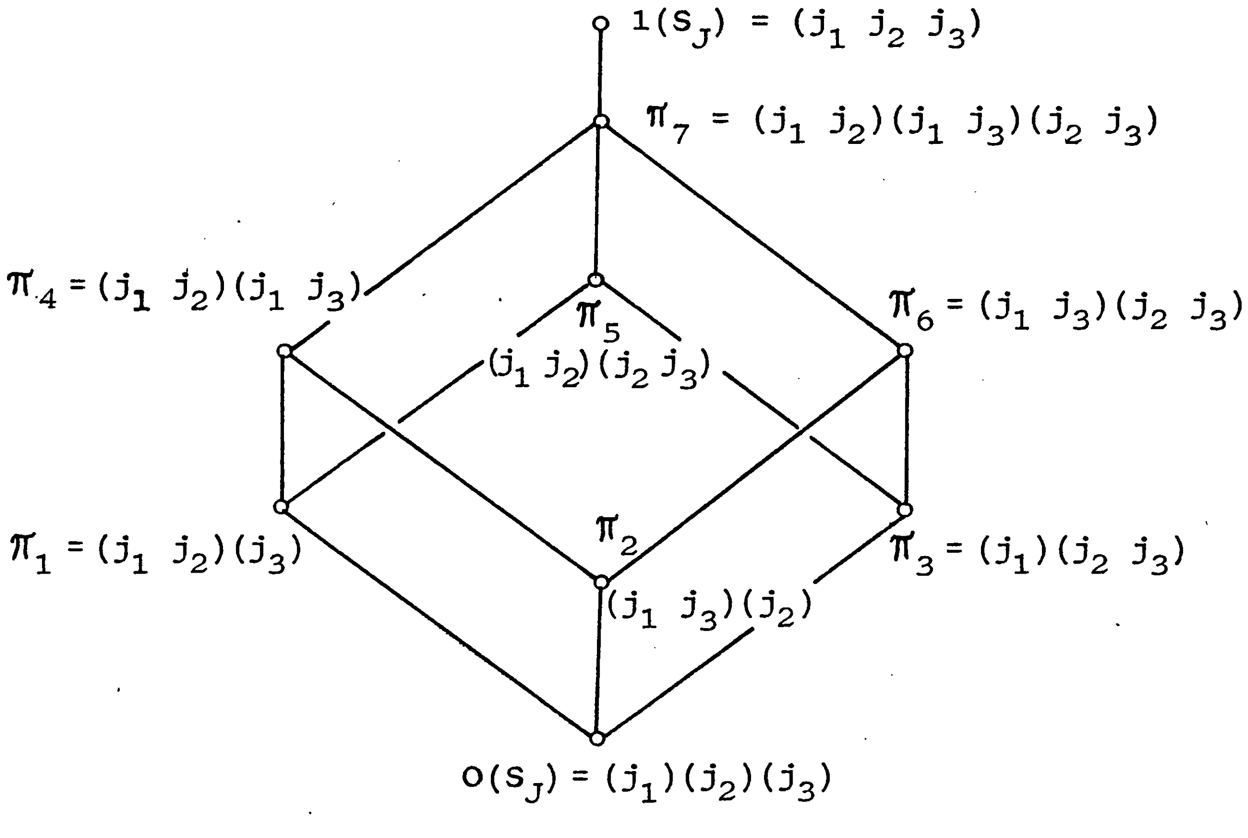
In the preceeding study of the product realisation, it was given at the outset that the one-many weak homomorphism θ relates the image F_1 to the stock semi-automaton P. Similarly, it was given that the one-many weak homomorphism $\mathcal{S}$ relates F_6 to the stock semiautomaton Q. In practice the designer has a complete "library" P, Q, R, ... of stock semiautomata, where each stock semiautomaton represents a sequential unit available from stock, and the designer has to assess these stock units as useful realisation components. That is, the designer must determine the one-many weak homomorphisms relating the images of the objective semiautomaton to the stock semiautomata P, Q, R, ..., and the corresponding stock units can then be regarded as potential components in a product realisation.

To assess the stock units in this way the designer can derive all the "nondegenerate" preserved covers of the objective semiautomaton, and can form the image semi-automata associated with each of these covers. By a "nondegenerate" cover it is meant that no cover block is a subset of another, for example $(j_1\ j_3)\ (j_2\ j_3)\ (j_2)$ is a "degenerate" cover of $S_J = \{j_1\ j_2\ j_3\}$ whereas the cover $\pi_6 = (j_1\ j_3)\ (j_2\ j_3)$ is nondegenerate. The nondegenerate preserved covers of a semiautomaton will form a "lattice" [Birkhoff], and this property can be used to derive the preserved covers using "meet" and "join" operators [Booth].

For example, the objective semiautomaton $J = \langle S_J \overline{X}_J \rangle$ from previously has the preserved-cover lattice of figure 6.9.

	x_1	x_2
j_1	j_2	j_3
j_2	-	j_3
j_3	j_2	j_3

(a) Objective semiautomaton $J = \langle S_J \overline{X}_J \rangle$



(b) Preserved-cover lattice of objective semiautomaton J

Figure 6.9

The upper universal bound is the "unity" cover $1(S_J) = (j_1\ j_2\ j_3)$, consisting of a single block containing all the objective states, and the lower universal bound is the "zero" cover $O(S_J) = (j_1)(j_2)(j_3)$ considered earlier.

The lattice provides a direct visualisation of the "mutually-resolving" preserved covers, for example the lattice shows that the preserved covers π_1 and π_6 are mutually-resolving since π_1 and π_6 have $O(S_J)$ as greatest lower bound. Consequently the associated image semiautomata are mutually-resolving, and this has been illustrated in figure 6.5, where the image semiautomata F_1 and F_6 based on these preserved covers gives a covering $J \lesssim F_1 \times F_6$ of the objective semiautomaton. Similarly the lattice shows that the preserved covers π_1 and π_3 are mutually-resolving, so a covering of J can be formed from associated image semiautomata, whereas π_4 and π_6 are not mutually-resolving since their greatest lower bound is π_2.

Having derived the preserved-cover lattice, the image semiautomata associated with each preserved cover can be formed and each image can be compared with the stock semiautomata, to try to find a one-many weak homomorphism.

For example the preserved cover $\pi_1 = (j_1\ j_2)(j_3)$ from the lattice defines the image semiautomaton $F_1 = \langle S_1\ \overline{X}_1 \rangle$ of figure 6.1, as reproduced in figure 6.10(a), and this image must be compared with each of the stock semiautomata P, Q, R,.... from the library.

$$(j_1\ j_2) \longleftarrow (j_3)$$

Mapping $\overline{x_1}^1$ over π_1

$$(j_1\ j_2) \longrightarrow (j_3)$$

Mapping $\overline{x_2}^1$ over π_1

	x_1	x_2
p_1	p_2	p_3
p_2	p_1	p_3
p_3	p_2	p_3
p_4	p_1	p_4

(a) Image semiautomaton $F_1 = \langle S_1\ \overline{X}_1 \rangle$

(b) Stock semiautomaton $P = \langle S_P\ \overline{X}_P \rangle$

Figure 6.10

Given the stock semiautomaton P of figure 6.10(b), the aim is to compare F_1 with semiautomaton P by seeking a one-many weak homomorphism of F_1 to P. The approach is to use implication trees, as detailed in chapter four, and to begin it is assumed that θ is a one-many weak homomorphism relating $(j_1\ j_2)$ from S_1 to p_1 from S_P, that is assume $\langle (j_1\ j_2)\ p_1 \rangle \in \theta$. Then the $\overline{x_1}^1$-successor $(j_1\ j_2)$ of $(j_1\ j_2)$ must relate to the $\overline{x_1}^P$-successor p_2 of p_1, which is valid since θ is one-many, and similarly the $\overline{x_2}^1$-successor (j_3) of $(j_1\ j_2)$ must relate to the $\overline{x_2}^P$-successor p_3 of p_1.

This gives level two of the implication tree of figure 6.11, and then forming level three gives entries in duplication of those from higher levels, so the tree is terminated without contradiction.

$$\begin{tikzcd}[column sep=small]
 & & & \langle (j_1\ j_2)\ p_1\rangle \in \theta \arrow[dll, Rightarrow] \arrow[drr, Rightarrow] & & & \\
 & \langle (j_1\ j_2)\ p_2\rangle \in \theta \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & & & \langle (j_3)\ p_3\rangle \in \theta \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\langle (j_1\ j_2)\ p_1\rangle & & \langle (j_3)\ p_3\rangle & & \langle (j_1\ j_2)\ p_2\rangle & & \langle (j_3)\ p_3\rangle
\end{tikzcd}$$

Figure 6.11

It remains to ensure that each block from $\pi_1 = (j_1\ j_2)\ (j_3)$ has been allocated to a state from S_P, and it can then be concluded that the relation

$$\theta = \{\langle (j_1\ j_2)\ p_1\rangle\ \langle (j_1\ j_2)\ p_2\rangle\ \langle (j_3)\ p_3\rangle\}$$

defined by the implication tree is a one-many weak homomorphism of F_1 to P. This relationship was illustrated previously as figure 6.1, and was used in forming the product realisation $J \leqslant P \times Q$ of figure 6.3.

This shows that product realisations can be investigated by finding the preserved covers of the objective semiautomaton, forming the image semiautomata associated with each preserved cover, and using implication trees to search for one-many weak homomorphisms relating these images to stock semiautomata. Such a weak homomorphism establishes a "realisation" of an image, and the realised images must then be considered to find a mutually-resolving family. However there may be many preserved covers of the objective semiautomaton, and each preserved cover will usually define several image semiautomata. Consequently this approach will often prove laborious, and would only be attractive if implemented by digital computer.

A more practical approach is to begin with the details of the stock semiautomata, and to use these details to form images of the objective semiautomaton so that each image is covered by some stock semiautomaton. These images can be formed directly from the details of the stock semiautomata, and the derivation of the preserved covers becomes unnecessary. For example the table represents the state transitions of a two-stage shift-register, and this can be formalised as a stock semiautomaton $R = \langle S_R \overline{X}_R \rangle$ where
$S_R = \{\langle 00 \rangle , \langle 01 \rangle , \langle 10 \rangle , \langle 11 \rangle\}$ and $X_R = \{0, 1\}$.

	0	1
$\langle 00 \rangle$	$\langle 00 \rangle$	$\langle 10 \rangle$
$\langle 01 \rangle$	$\langle 00 \rangle$	$\langle 10 \rangle$
$\langle 10 \rangle$	$\langle 01 \rangle$	$\langle 11 \rangle$
$\langle 11 \rangle$	$\langle 01 \rangle$	$\langle 11 \rangle$

Stock Semiautomaton

$R = \langle S_R \overline{X}_R \rangle$

To begin, it is important to consider the way R relates to the objective semiautomaton J. If semiautomaton J is covered by R, the shift register gives a "simple" realisation

$\hat{J} \leqslant \hat{R}$ of the objective automaton and a product realisation need not be attempted. This can be checked as detailed in chapter four, by using implication trees to seek an input assignment α of X_J to X_R and a state assignment γ relating S_J to S_R, so that γ is a one-many weak homomorphism of J to R under α. Here, however, this will verify that no assignment $\langle \alpha \ \gamma \rangle$ of J into R exists, so the shift register cannot be used to give a simple realisation of the objective automaton.

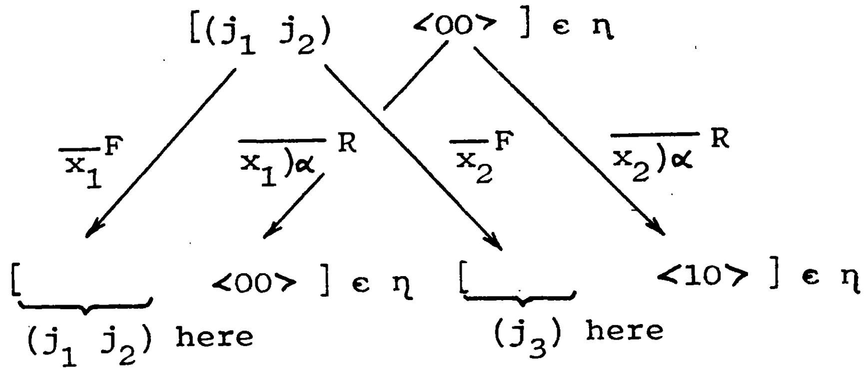
Then the aim is to use the details of stock semiautomaton R to form an image F of objective semiautomaton J, so that F is related to R by a one-many weak homomorphism under an appropriate input assignment. It will usually be useful to form a "power successor" table for the objective semiautomaton, showing the image $(B)\overline{x}^J$ of each subset $B \subseteq S_J$ under each of the transition mappings $\overline{x}^J$. For example the table shows that the image of the subset $\{j_1\ j_3\} \subseteq S_J$ under $\overline{x_1}^J$ is $\{j_2\}$, since $\langle j_1\ j_2 \rangle \in \overline{x_1}^J$ and $\langle j_3\ j_2 \rangle \in \overline{x_1}^J$, and similarly the image of $\{j_1\ j_3\}$ under $\overline{x_2}^J$ is $\{j_3\}$.

$B \subseteq S_J$	$(B)\,\overline{x_1}^J$	$(B)\,\overline{x_2}^J$
$\{j_1\}$	$\{j_2\}$	$\{j_3\}$
$\{j_2\}$	$\emptyset$	$\{j_3\}$
$\{j_3\}$	$\{j_2\}$	$\{j_3\}$
$\{j_1\ j_2\}$	$\{j_2\}$	$\{j_3\}$
$\{j_1\ j_3\}$	$\{j_2\}$	$\{j_3\}$
$\{j_2\ j_3\}$	$\{j_2\}$	$\{j_3\}$
$\{j_1\ j_2\ j_3\}$	$\{j_2\}$	$\{j_3\}$

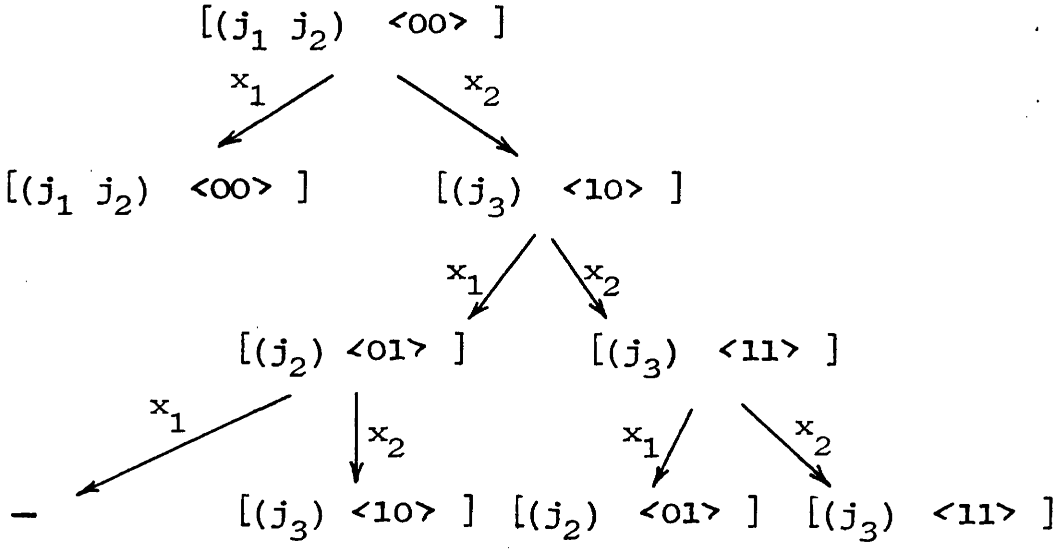
<u>Power successor table for semiautomaton J</u>

In this case the image of each subset of S_J is a singleton, but in general the entries will be various subsets of the objective automaton state-set.

Consider now the subsets of S_J containing just two elements, and assume that one of these subsets, for example $\{j_1\ j_2\}$, forms a block $(j_1\ j_2)$ of a S_J-cover S_F. Furthermore assume $F = \langle S_F\ \overline{X}_F \rangle$ is an image semiautomaton of J, define $\alpha = \{\langle x_1\ \ 0\rangle\ \ \ \langle x_2\ \ 1\rangle\}$, and assume that η is a one-many weak homomorphism of image semiautomaton F to R under α, where the block $(j_1\ j_2)$ from S_F is assigned by η to $\langle 00\rangle$ from S_R. This assumption $[(j_1\ j_2)\ \ \langle 00\rangle\ ] \in \eta$ is represented in figure 6.12(a), and then since η is a weak homomorphism under α the $\overline{x_1}^F$-successor of $(j_1\ j_2)$ must relate to the $\overline{x_1)\alpha}^{\,R} = \overline{0}^R$ successor of $\langle 00\rangle$. That is, the $\overline{x_1}^F$-successor of $(j_1\ j_2)$ must relate under η to $\langle 00\rangle$. Furthermore $F = \langle S_F\ \overline{X}_F\rangle$ is an image semiautomaton of J, so the $\overline{x_1}^F$-successor of $(j_1\ j_2)$ must have the $\overline{x_1}^J$-image of block $(j_1\ j_2)$ as a subset. From the power successor table the $\overline{x_1}^J$-image of $\{j_1\ j_2\}$ is $\{j_2\}$, so the $\overline{x_1}^F$-successor of $(j_1\ j_2)$ must have $\{j_2\}$ as a subset. This must now be reconciled with preceeding assumptions, and agreement can be achieved here by defining $(j_1\ j_2)$ to be the $\overline{x_1}^F$-successor of $(j_1\ j_2)$. Similarly η must relate the $\overline{x_2}^F$-successor of $(j_1\ j_2)$ to the $\overline{x_2)\alpha}^{\,R} = \overline{1}^R$ successor $\langle 10\rangle$ of $\langle 00\rangle$, furthermore the $\overline{x_2}^F$-successor of $(j_1\ j_2)$ must have the $\overline{x_2}^J$-image $\{j_3\}$ of $\{j_1\ j_2\}$ as a subset. These requirements are satisfied by defining



(a) Implications of $[(j_1\ j_2)\quad \langle 00\rangle\ ] \in \eta$



(b) Completed implication tree

Figure 6.12

(j_3) to be the $\overline{x_2}^F$-successor of $(j_1\ j_2)$, and setting $[(j_3)\quad \langle 10\rangle\ ] \in \eta$.

Now the implications of the assignment $[(j_3)\quad \langle 10\rangle\ ] \in \eta$ must be considered, and this is shown in the implication tree of figure 6.12(b). The $\overline{x_1}^J$-image of $\{j_3\}$ is $\{j_2\}$, consequently $\{j_2\}$ must be a subset of the $\overline{x_1}^F$-successor of (j_3), furthermore the $\overline{x_1}^F$-successor of (j_3) must relate under η to $\langle 01\rangle$. This can be satisfied by defining (j_2) to be the

$\overline{x}_1^F$-successor of (j_3), and setting $[(j_2) \ \langle 01 \rangle \ ] \in \eta$. Similarly the $\overline{x}_2^F$-successor of (j_3) must have $\{j_3\}$ as a subset, and must relate under η to $\langle 11 \rangle$, and this can be satisfied by defining (j_3) to be the $\overline{x}_2^F$-successor of (j_3) and setting $[(j_3) \ \langle 11 \rangle \ ] \in \eta$. Then the implications of the assignments $[(j_2) \ \langle 01 \rangle \ ] \in \eta$ and $[(j_3) \ \langle 11 \rangle \ ] \in \eta$ must be traced, and the tree must be abandoned if a requirement cannot be reconciled with the assumption that η is one-many, and that $F = \langle S_F \ \overline{X}_F \rangle$ is an image semiautomaton.

In the present example, however, the tree can be terminated without contradiction, since continuing the tree repeats assignments on higher levels. The implication tree of figure 6.12(b) gives

$$\overline{x}_1^F = \{\langle (j_1 \ j_2) \ (j_1 \ j_2) \rangle \ \ \langle (j_3) \ (j_2) \rangle\} \text{ and}$$

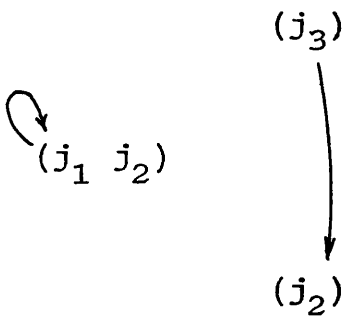
$$\overline{x}_2^F = \{\langle (j_1 \ j_2) \ (j_3) \rangle \ \ \langle (j_2) \ (j_3) \rangle \ \ \langle (j_3) \ (j_3) \rangle\},$$

giving the image semiautomaton $F = \langle S_F \ \overline{X}_F \rangle$ shown in figure 6.13(a), with $S_F = (j_1 \ j_2)(j_2)(j_3)$ and $\overline{X}_F = \{\overline{x}_1^F, \overline{x}_2^F\}$.

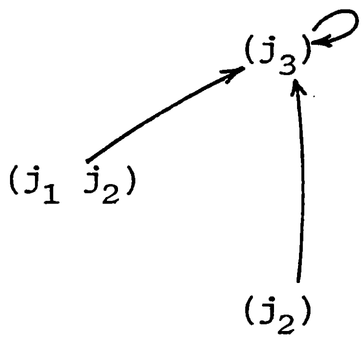
Furthermore the implication tree gives the relation η from S_F to S_R, where

$$\eta = \left\{ \begin{matrix} [(j_1 \ j_2) \ \ \langle 00 \rangle \ ][(j_3) \ \ \langle 10 \rangle \ ] \\ [(j_2) \ \ \langle 01 \rangle \ ][(j_3) \ \ \langle 11 \rangle \ ] \end{matrix} \right\},$$

and this is a one-many weak homomorphism of image semiautomaton F to the stock semiautomaton R under the input assignment α, as shown in figure 6.13.



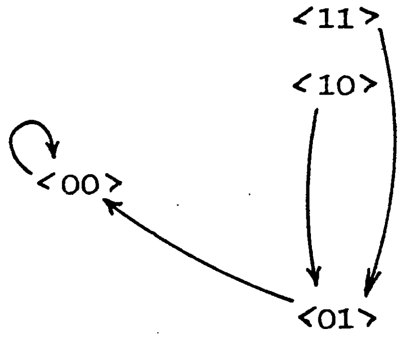
Mapping $\overline{x_1}^F$ over S_F



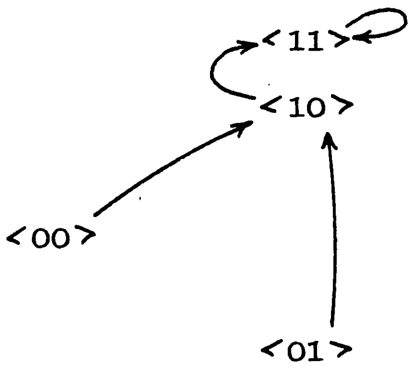
Mapping $\overline{x_2}^F$ over S_F

<u>(a) Image semiautomaton</u>

$\underline{F = \langle S_F \overline{X}_F \rangle}$



Mapping $\overline{x_1)\alpha}^{\,R} = \overline{0}^R$ over S_R



Mapping $\overline{x_2)\alpha}^{\,R} = \overline{1}^R$ over S_R

<u>(b) Stock semiautomaton</u>

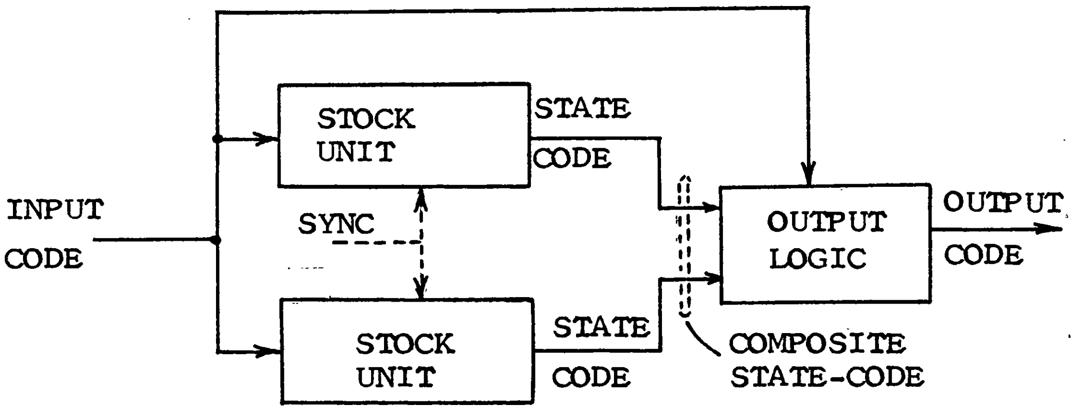
$\underline{R = \langle S_R \overline{X}_R \rangle}$

<u>Figure 6.13</u>

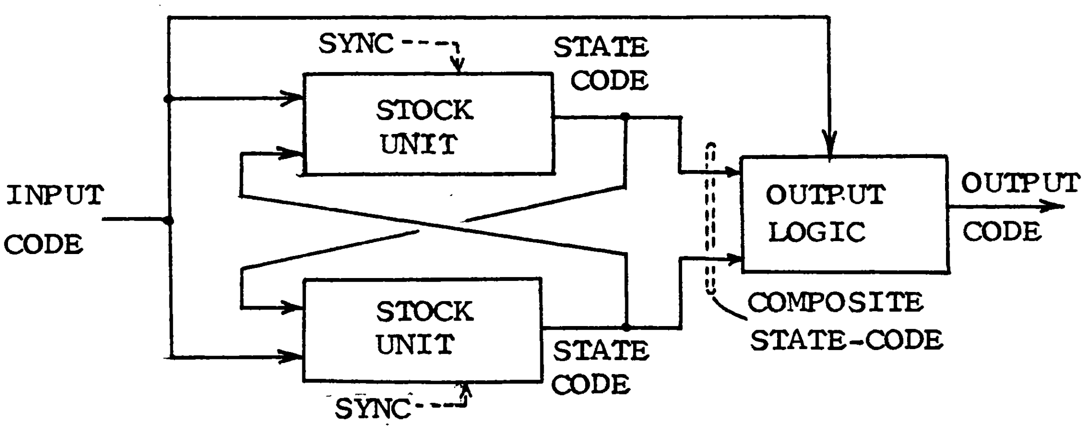
This shows that implication trees can be used to derive the images covered by given stock semiautomata. The approach can be used to form the image semiautomata covered by stock semiautomata from the library P, Q, R,...., and the "realised" image semiautomata must then be considered to find a mutually-resolving family. For example from above $F \leqslant R$, so a covering $F' \leqslant R'$, where R' is a stock semiautomaton and F, F' are mutually-resolving images of J, would give a covering $J \leqslant F \times F' \leqslant R \times R'$ and this would give a corresponding product realisation of the objective automaton $\hat{J}$. The present aim is to consider the approach should just one image covering be found. For example, given $F \leqslant R$ as above, it may be that no realisation $F' \leqslant R'$ of a "complementary" image F' by a stock semiautomaton R' exists. The approach is to search for a stock unit to be "cascaded" with the stock unit R, to give a "cascade realisation" of the objective automaton.

6.4 Cascade realisations

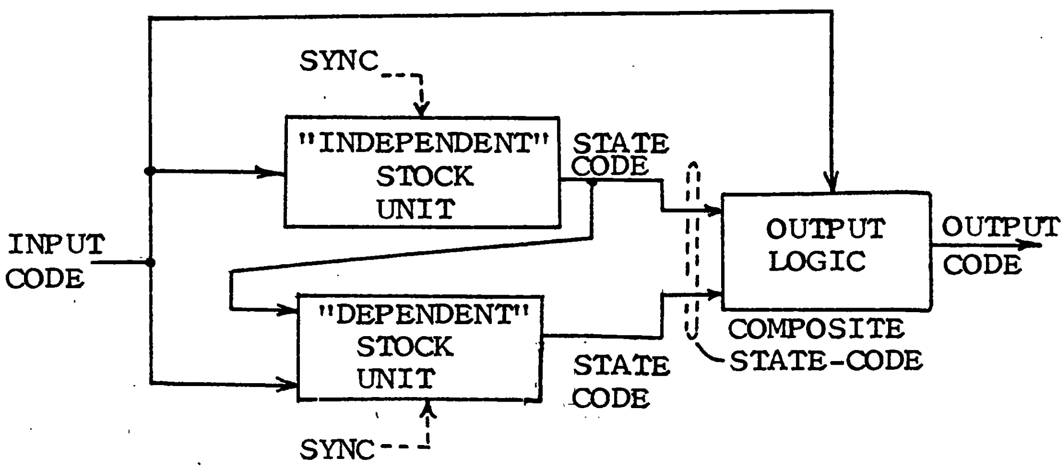
The important feature of the product construction, giving realisations in the form of figure 6.14(a), is the independence of the component units. Clearly the operation of each unit depends entirely on the applied input code, and is independent of the operation of the other. This contrasts with the scheme of figure 6.14(b), which shows a fully-interactive interconnection of stock units. Here the state-code of each unit is supplied as input to the other, and combines with the applied input code to influence state transitions. In the simpler construction of figure 6.14(c) one of the units is independent of the other, but the second is "dependent", since the present state of the first unit combines with the applied input code to form the input condition for the second. These units are connected in "cascade", and the state transitions of the composite system are expressed by the cascade product of the semiautomata representing the component units.



(a) Product construction



(b) Interdependent stock units



(c) Stock units in cascade

Figure 6.14

For example let the semiautomata $A = \langle S_A\ \overline{X_A} \rangle$ and $B = \langle S_B\ \overline{X_B} \rangle$ represent stock units, the details of these stock semiautomata being given in the tables of figure 6.15. Then from these tables $X_A = \{x_1, x_2\}$, $S_A = \{a_1, a_2, a_3, a_4\}$ and $X_B = X_A \times S_A$, in which case the units can be connected in cascade as in figure 6.15(c). In general such a composite system can be formed whenever $X_A \times S_A \subseteq X_B$, giving a composite system with input set X_A. The state transitions of the composite system are then expressed by the composite semiautomaton $C = A \circ B$, the "cascade" of semiautomaton A with semiautomaton B. Define $C = \langle S_C\ \overline{X_C} \rangle$ where $C = A \circ B$, in which case $S_C = S_A \times S_B$, $X_C = X_A$, and $x \in X_C$ implies $\overline{x}^C \in \overline{X_C}$ where

$$\overline{x}^C = \left\{ [\, \langle a\ b \rangle\ \langle a'\ b' \rangle \,] \;\middle|\; \langle a\ a' \rangle \in \overline{x}^A,\ \langle b\ b' \rangle \in \overline{\langle x\ a \rangle}^B \right\}$$

For example table 6.15(a) shows $\langle a_1\ a_2 \rangle \in \overline{x_1}^A$, and table 6.15(b) shows $\langle b_1\ b_2 \rangle \in \overline{\langle x_1\ a_1 \rangle}^B$ so $[\, \langle a_1\ b_1 \rangle\ \ \langle a_2\ b_2 \rangle \,] \in \overline{x_1}^C$. Hence the composite state changes from $\langle a_1\ b_1 \rangle$ to $\langle a_2\ b_2 \rangle$ in response to input x_1, since a_2 is the $\overline{x_1}^A$-successor of a_1 and b_2 is the $\overline{\langle x_1\ a_1 \rangle}^B$-successor of b_1. Continuing this reasoning gives the mappings $\overline{x_1}^C$ and $\overline{x_2}^C$ over $S_C = S_A \times S_B$, as expressed in table 6.15(d), and this completes the formalisation of the composite semiautomaton $A \circ B = C = \langle S_C\ \overline{X_C} \rangle$.

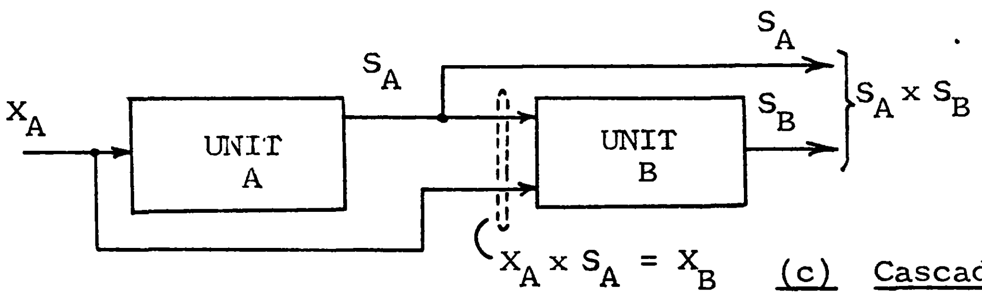
Then the cascade semiautomaton $C = \langle S_C\ \overline{X_C} \rangle$ is a realisation of the objective semiautomaton $J = \langle S_J\ \overline{X_J} \rangle$, from previously, and can be used as the basis of a cascade realisation of the objective automaton $\hat{J}$.

S_A	x_1	x_2
a_1	a_2	a_3
a_2	a_1	a_3
a_3	a_3	a_3
a_4	a_3	a_2

(a) Stock semiautomaton $A = \langle S_A \overline{X}_A \rangle$

S_B	$\langle x_1\ a_1 \rangle$	$\langle x_1\ a_2 \rangle$	$\langle x_1\ a_3 \rangle$	$\langle x_1\ a_4 \rangle$	$\langle x_2\ a_1 \rangle$	$\langle x_2\ a_2 \rangle$	$\langle x_2\ a_3 \rangle$	$\langle x_2\ a_4 \rangle$
b_1	b_2	b_2	b_2	b_1	b_1	b_1	b_1	b_2
b_2	b_2	b_1	b_1	b_2	b_1	b_1	b_1	b_1

(b) Stock semiautomaton $B = \langle S_B \overline{X}_B \rangle$



(c) Cascade interconnection

	x_1	x_2
$\langle a_1\ b_1 \rangle$	$\langle a_2\ b_2 \rangle$	$\langle a_3\ b_1 \rangle$
$\langle a_2\ b_1 \rangle$	$\langle a_1\ b_2 \rangle$	$\langle a_3\ b_1 \rangle$
$\langle a_3\ b_1 \rangle$	$\langle a_3\ b_2 \rangle$	$\langle a_3\ b_1 \rangle$
$\langle a_4\ b_1 \rangle$	$\langle a_3\ b_1 \rangle$	$\langle a_2\ b_2 \rangle$
$\langle a_1\ b_2 \rangle$	$\langle a_2\ b_2 \rangle$	$\langle a_3\ b_1 \rangle$
$\langle a_2\ b_2 \rangle$	$\langle a_1\ b_1 \rangle$	$\langle a_3\ b_1 \rangle$
$\langle a_3\ b_2 \rangle$	$\langle a_3\ b_1 \rangle$	$\langle a_3\ b_1 \rangle$
$\langle a_4\ b_2 \rangle$	$\langle a_3\ b_2 \rangle$	$\langle a_2\ b_1 \rangle$

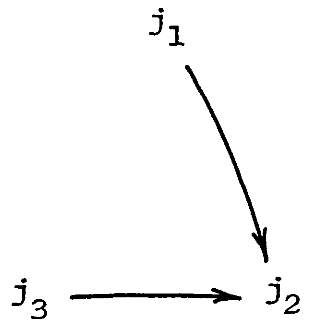
(d) Semiautomaton $C = \langle S_C \overline{X}_C \rangle$

Figure 6.15

This is evident from the figures 6.16(a) and 6.16(b), where a one-many relation

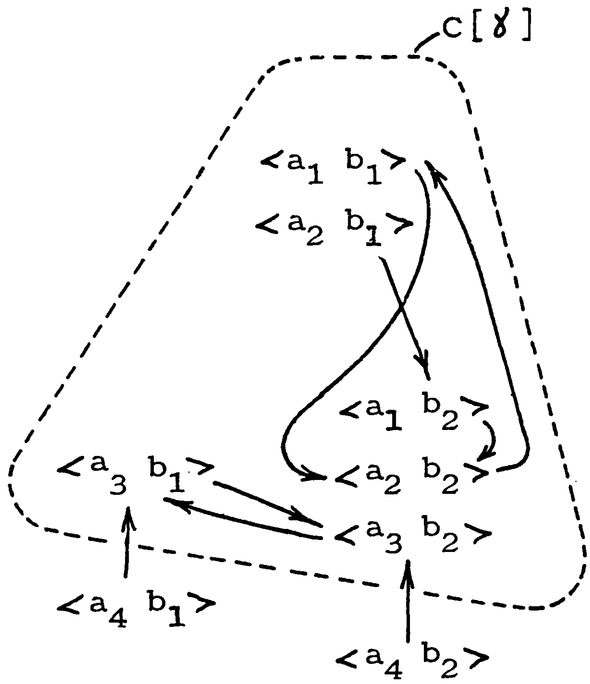
$$\gamma = \left\{\begin{matrix} [j_1 \ \langle a_1 \ b_1\rangle\ ][j_2 \ \langle a_1 \ b_2\rangle\ ][j_2 \ \langle a_3 \ b_2\rangle\ ] \\ [j_1 \ \langle a_2 \ b_1\rangle\ ][j_2 \ \langle a_2 \ b_2\rangle\ ][j_3 \ \langle a_3 \ b_1\rangle\ ] \end{matrix}\right\}$$

from S_J to $S_A \times S_B$ is expressed implicitly, by arranging the codomain $C[\gamma]$ accordingly. The mappings $\overline{x_1}^C$ and $\overline{x_2}^C$ are those expressed by table 6.15(d), and the figure shows that γ is a one-many weak homomorphism of J to C = A o B, that is $J \leqslant^{\gamma} A \circ B$. For example figure 6.16(a) shows $[j_1 \ \langle a_1 \ b_1\rangle\ ] \in \gamma$, that is $[\langle a_1 \ b_1\rangle \ j_1] \in \gamma^{-1}$, and the figure shows $\langle j_1 \ j_2\rangle \in \overline{x_1}^J$ so $[\langle a_1 \ b_1\rangle \ j_2] \in \gamma^{-1}\overline{x_1}^J$. Furthermore figure 6.16(a) shows $[\langle a_1 \ b_1\rangle \ \langle a_2 \ b_2\rangle\ ] \in \overline{x_1}^C$ and $[\langle a_2 \ b_2\rangle \ j_2] \in \gamma^{-1}$, so $[\langle a_1 \ b_1\rangle \ j_2] \in \overline{x_1}^C \gamma^{-1}$, in accordance with the inclusion $\gamma^{-1}\overline{x_1}^J \subseteq \overline{x_1}^C \gamma^{-1}$. Continuing this reasoning confirms $\gamma^{-1}\overline{x_1}^J \subseteq \overline{x_1}^C \gamma^{-1}$, and similarly figure 6.16(b) shows $\gamma^{-1}\overline{x_2}^J \subseteq \overline{x_2}^C \gamma^{-1}$, furthermore γ has S_J as domain so γ is a weak homomorphism of J to A o B. Here the codomain $C[\gamma]$ forms a sub-algebra of $C = \langle S_C \ \overline{X}_C\rangle$, but this will not be the case in general, since the image of a partial semiautomaton under a weak homomorphism is not always a subalgebra. Finally γ is one-many, confirming $J \leqslant^{\gamma} A \circ B$, so the stock units can be used to give a cascade realisation of automaton $\hat{J}$ as in figure 6.14(c), with stock unit A "independent" and stock unit B "dependent".



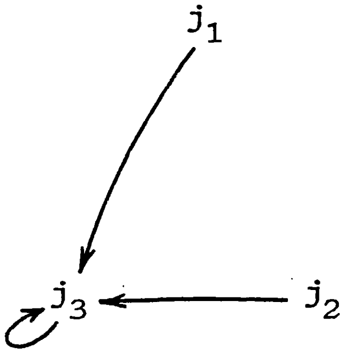
Mapping $\overline{x_1}^J$ over S_J

$\xrightarrow{\gamma}$



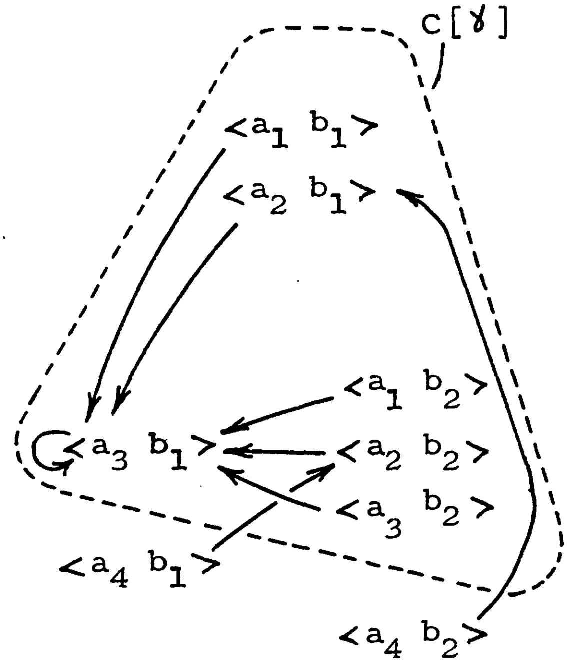
Mapping $\overline{x_1}^C$ over $S_C = S_A \times S_B$

(a) $\gamma^{-1}\overline{x_1}^J \subseteq \overline{x_1}^C \gamma^{-1}$



Mapping $\overline{x_2}^J$ over S_J

$\xrightarrow{\gamma}$



Mapping $\overline{x_2}^C$ over $S_C = S_A \times S_B$

(b) $\gamma^{-1}\overline{x_2}^J \subseteq \overline{x_2}^C \gamma^{-1}$

Figure 6.16

To develop a systematic approach to cascade realisation, consider the table of figure 6.17(a) giving objective semiautomaton $J = \langle S_J \overline{X}_J \rangle$ from previously.

	x_1	x_2
j_1	j_2	j_3
j_2	-	j_3
j_3	j_2	j_3

(a) <u>Objective semiautomaton $J = \langle S_J \overline{X}_J \rangle$</u>

$$\circlearrowright (j_1\ j_2) \qquad\qquad (j_2\ j_3) \circlearrowleft$$

Mapping $\overline{x}_1^F$ over $S_F = \pi$

$$(j_1\ j_2) \longrightarrow (j_2\ j_3) \circlearrowleft$$

Mapping $\overline{x}_2^F$ over $S_F = \pi$

(b) <u>π-image semiautomaton $F = \langle S_F \overline{X}_F \rangle$</u>

<u>Figure 6.17</u>

Then $\pi = (j_1\ j_2)(j_2\ j_3)$ is a J-preserved S_J-cover, and this is shown in the lattice of figure 6.9 since $\pi = \pi_5$. Consequently at least one π-image of $J = \langle S_J \overline{X}_J \rangle$ can be formed, and figure 6.17(b) shows a π-image semiautomaton $F = \langle S_F \overline{X}_F \rangle$ where $S_F = \pi$, $X_F = X_J$ and $\overline{X}_F = \{\overline{x}_1^F, \overline{x}_2^F\}$.

Such an image semiautomaton can be used to form a cascade-composite semiautomaton covering J, since there must be at least one semiautomaton R so that $J \leqslant F \circ R$. To formalise such a semiautomaton R, attention is directed to S_J-covers τ satisfying $\pi * \tau = 0(S_J) = (j_1)(j_2)(j_3)$. Any choice of a

S_J-cover τ so that $\pi * \tau = O(S_J)$ will be satisfactory, for example $\pi = (j_1\ j_2)(j_2\ j_3)$ so define $\tau = (j_1\ j_3)(j_2)$, and then the intersection table verifies $\pi * \tau = O(S_J)$.

		τ: $(j_1\ j_3)$	τ: (j_2)
π	$(j_1\ j_2)$	$\{j_1\}$	$\{j_2\}$
π	$(j_2\ j_3)$	$\{j_3\}$	$\{j_2\}$

<u>Intersection table showing</u>
<u>$\pi * \tau = O(S_J)$</u>

It is emphasised that τ can be any S_J-cover satisfying $\pi * \tau = O(S_J)$, although in the present example $\tau = (j_1\ j_3)(j_2)$ is a partition. In particular the S_J-cover τ need not be preserved, however $\tau = (j_1\ j_3)(j_2)$ is a preserved cover of semiautomaton J, indeed in the present example any S_J-cover must be preserved since $(S_J)\overline{x_1}^J$ and $(S_J)\overline{x_2}^J$ are singletons.

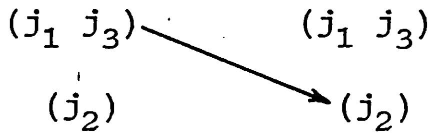
The S_J-cover $\tau = (j_1\ j_3)(j_2)$ can now be used to formalise a semiautomaton R, so that $J \leqslant F \circ R$. Define $R = \langle S_R\ \overline{X_R} \rangle$ where $S_R = \tau$, define $X_R = X_F \times S_F$, and associate with each pair $\langle x\ f \rangle \in X_F \times S_F$ a mapping $\overline{\langle x\ f \rangle}^R$ over τ, defined as follows. For any $r \in \tau$, if $(f \cap r)\overline{x}^J = \emptyset$ then r need not be appointed a $\overline{\langle x\ f \rangle}^R$-successor, that is r can be excluded from the domain $D[\overline{\langle x\ f \rangle}^R]$. However if $(f \cap r)\overline{x}^J \neq \emptyset$ some $\overline{\langle x\ f \rangle}^R$-successor for r must be appointed, and must be a block r' from τ satisfying $(f \cap r)\overline{x}^J \subseteq r'$. Assuming $(f \cap r)\overline{x}^J \neq \emptyset$ then $(f \cap r)\overline{x}^J$ must be a singleton, since $\pi * \tau = 0(S_J)$ ensures that $f \cap r$ is a singleton if nonvoid, and $\overline{x}^J$ is a mapping so $(f \cap r)\overline{x}^J$ must be a singleton if nonvoid. Furthermore τ is a S_J-cover, so there must be at least one block r' of cover τ with the singleton $(f \cap r)\overline{x}^J$ as a subset, that is there must be at least one block r' of cover τ satisfying $(f \cap r)\overline{x}^J \subseteq r'$. In general there will be several blocks r' of cover τ so that $(f \cap r)\overline{x}^J \subseteq r'$, and any of these candidates can be taken as the $\overline{\langle x \; f \rangle}^R$-successor for r.

For the present example, where $S_F = \pi = (j_1 \; j_2)(j_2 \; j_3)$ and $X_F = X_J = \{x_1, x_2\}$, the Cartesian product $X_F \times S_F$ is given by

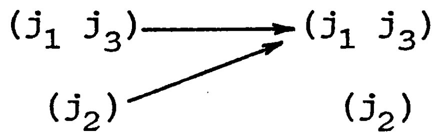
$$X_F \times S_F = \left\{ \begin{matrix} \langle x_1 \quad (j_1 \; j_2) \rangle & \langle x_2 \quad (j_1 \; j_2) \rangle \\ \langle x_1 \quad (j_2 \; j_3) \rangle & \langle x_2 \quad (j_2 \; j_3) \rangle \end{matrix} \right\}$$

so four mappings over τ must be formalised. Consider first the mapping $\overline{\langle x_1 \; (j_1 \; j_2) \rangle}^R$, and consider the appointment of a $\overline{\langle x_1 \; (j_1 \; j_2) \rangle}^R$-successor for the block $(j_1 \; j_3)$ from τ. Putting $f = (j_1 \; j_2)$ and $r = (j_1 \; j_3)$ gives $f \cap r = \{j_1\}$, furthermore $\langle j_1 \; j_2 \rangle \in \overline{x_1}^J$ so $(f \cap r)\overline{x_1}^J = \{j_2\}$. Clearly $(f \cap r)\overline{x_1}^J \neq \emptyset$, so a successor r' must be appointed and must satisfy $(f \cap r)\overline{x_1}^J \subseteq r'$, that is r' must satisfy $\{j_2\} \subseteq r'$. In general there would be several possible choices for r', however $\tau = (j_1 \; j_3)(j_2)$ has just one block with subset $\{j_2\}$, so block (j_2) is the only candidate as the $\overline{\langle x_1 \; (j_1 \; j_2) \rangle}^R$-successor for $(j_1 \; j_3)$, that is set

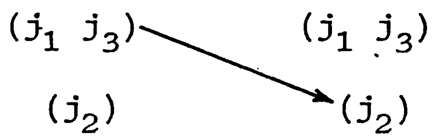
$\langle(j_1\ j_3)(j_2)\rangle \in \overline{\langle x_1\ (j_1\ j_2)\rangle}^R$. Similarly, consider the appointment of a $\overline{\langle x_1\ (j_1\ j_2)\rangle}^R$-successor for the block (j_2) from τ. Putting $f = (j_1\ j_2)$ and $r = (j_2)$ gives $f \cap r = \{j_2\}$, however j_2 has no $\overline{x_1}^J$-successor, so $(f \cap r)\overline{x}_1^J = \emptyset$. Consequently no $\overline{\langle x_1\ (j_1\ j_2)\rangle}^R$-successor for block (j_2) is appointed, and $\overline{\langle x_1\ (j_1\ j_2)\rangle}^R$ is the mapping shown in figure 6.18(a).



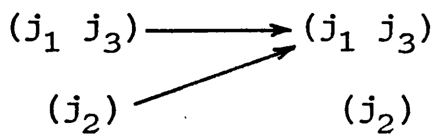
(a) Mapping $\overline{\langle x_1\ (j_1\ j_2)\rangle}^R$ over τ



(c) Mapping $\overline{\langle x_2\ (j_1\ j_2)\rangle}^R$ over τ



(b) Mapping $\overline{\langle x_1\ (j_2\ j_3)\rangle}^R$ over τ



(d) Mapping $\overline{\langle x_2\ (j_2\ j_3)\rangle}^R$ over τ

Figure 6.18

Continuing this reasoning gives the remaining mappings over τ as in figure 6.18, one mapping for each element from $X_F \times S_F$, and this concludes the formalisation of the semiautomaton $R = \langle S_R\ \overline{X}_R\rangle$.

Considering now the cascade-composite semiautomaton $F \circ R$ define $K = \langle S_K\ \overline{X}_K\rangle$ where $K = F \circ R$, so $S_K = S_F \times S_R$, $S_F \times S_R = \pi \times \tau$ and $X_K = X_F = \{x_1, x_2\}$. Then $x \in X_K$ implies

$\overline{x}^K \in \overline{X}_K$ where

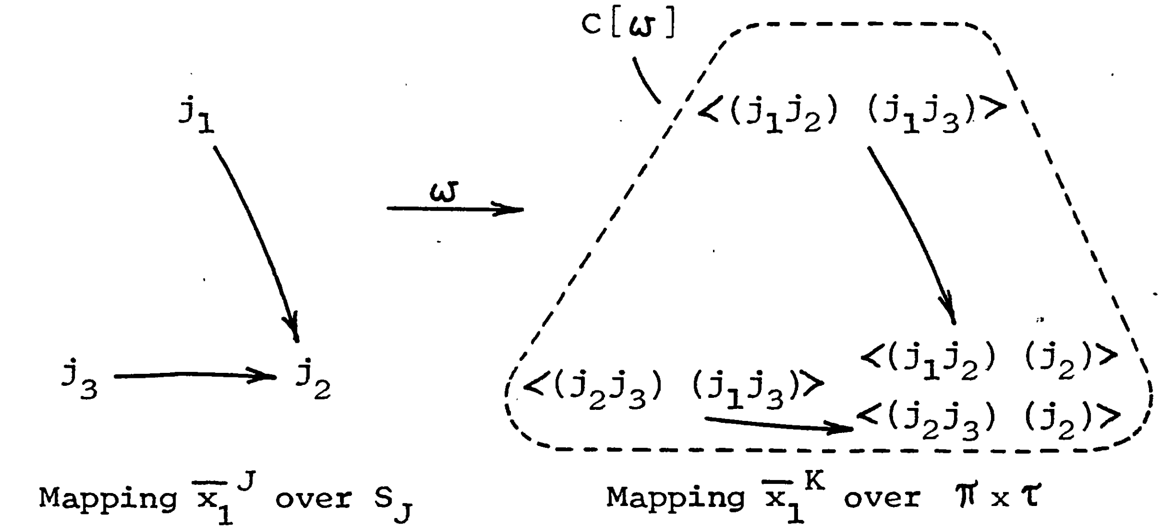
$\overline{x}^K = \left\{ [\ \langle f\ r\rangle \quad \langle f'\ r'\rangle\ ]\ \middle|\ \langle f\ f'\rangle \in \overline{x}^F\ \&\ \langle r\ r'\rangle \in \overline{\langle x\ f\rangle}^R \right\}$,

for example figure 6.17(b) shows $\langle (j_1\ j_2)\ (j_1\ j_2)\rangle \in \overline{x_1}^F$ and figure 6.18(a) shows $\langle (j_1\ j_3)\ (j_2)\rangle \in \overline{\langle x_1\ (j_1\ j_2)\rangle}^R$ so $[\ \langle (j_1\ j_2)(j_1\ j_3)\rangle \quad \langle (j_1\ j_2)\ (j_2)\rangle\ ] \in \overline{x_1}^K$. This gives the mappings $\overline{x_1}^K$ and $\overline{x_2}^K$ over $\pi \times \tau$, as expressed in the table defining semiautomaton $K = \langle S_K\ \overline{X}_K\rangle$.

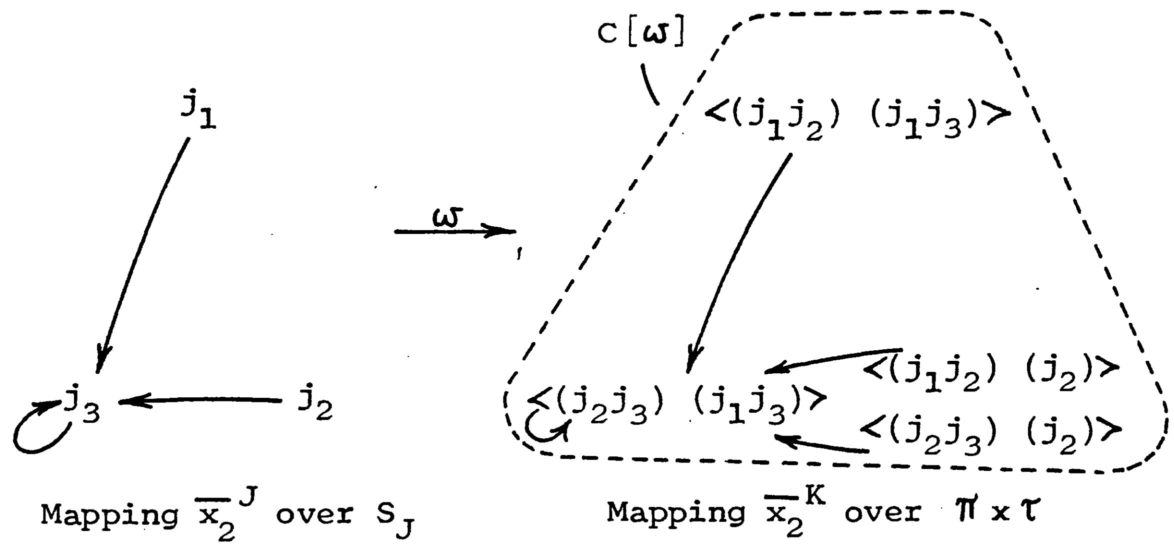
	x_1	x_2
$\langle (j_1\ j_2)\ (j_1\ j_3)\rangle$	$\langle (j_1\ j_2)\ (j_2)\rangle$	$\langle (j_2\ j_3)\ (j_1\ j_3)\rangle$
$\langle (j_1\ j_2)\ (j_2)\rangle$	-	$\langle (j_2\ j_3)\ (j_1\ j_3)\rangle$
$\langle (j_2\ j_3)\ (j_1\ j_3)\rangle$	$\langle (j_2\ j_3)\ (j_2)\rangle$	$\langle (j_2\ j_3)\ (j_1\ j_3)\rangle$
$\langle (j_2\ j_3)\ (j_2)\rangle$	-	$\langle (j_2\ j_3)\ (j_1\ j_3)\rangle$

Semiautomaton $K = \langle S_K\ \overline{X}_K\rangle$, where $K = F \circ R$

Then $J \leqslant F \circ R$, and this is evident from figure 6.19. The mappings $\overline{x_1}^K$ and $\overline{x_2}^K$ over $\pi \times \tau$ are those expressed by the table, and the relationship between J and $F \circ R$ can be appreciated by considering an arbitrary objective state $j \in S_J$. Since π is a S_J-cover there must be at least one block f from π where $j \in f$, similarly there must be at least one block r of S_J-cover τ where $j \in r$, so there must be at least one pair $\langle f\ r\rangle \in \pi \times \tau$ where $j \in f$ and $j \in r$.



(a) Showing $\omega^{-1}\overline{x}_1^J \subseteq \overline{x}_1^K \omega^{-1}$



(b) Showing $\omega^{-1}\overline{x}_2^J \subseteq \overline{x}_2^K \omega^{-1}$

Figure 6.19

This correspondence between S_J and $\pi \times \tau$ can be formalised as a relation ϖ where

$$\varpi = \left\{ [j \quad \langle f \; r \rangle] \,\middle|\, j \in S_J, \langle f \; r \rangle \in \pi \times \tau, j \in f \;\; \& \;\; j \in r \right\}$$

and for the example $\pi = (j_1 \; j_2)(j_2 \; j_3)$ and

$\tau = (j_1 \; j_3)(j_2)$ so

$$\pi \times \tau = \left\{ \begin{matrix} \langle (j_1 \; j_2) \; (j_1 \; j_3) \rangle & \langle (j_2 \; j_3) \; (j_1 \; j_3) \rangle \\ \langle (j_1 \; j_2) \; (j_2) \rangle & \langle (j_2 \; j_3) \; (j_2) \rangle \end{matrix} \right\}$$

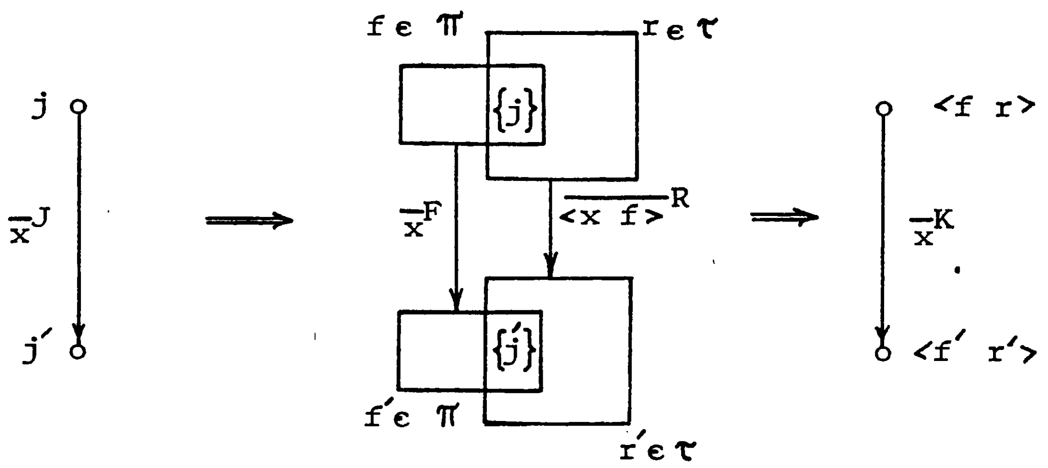
giving

$$\varpi = \left\{ \begin{matrix} [j_1 \quad \langle (j_1 \; j_2) \; (j_1 \; j_3) \rangle \,] [j_3 \quad \langle (j_2 \; j_3) \; (j_1 \; j_3) \rangle \,] \\ [j_2 \quad \langle (j_1 \; j_2) \; (j_2) \rangle \,] [j_2 \quad \langle (j_2 \; j_3) \; (j_2) \rangle \,] \end{matrix} \right\}$$

Clearly ϖ is one-many with S_J as domain, and it is evident from figure 6.19 that ϖ is a one-many weak homomorphism of J to F o R, that is $J \leqslant^{\varpi} F \circ R$. Here ϖ is expressed implicitly, for example figure 6.19(a) shows $[j_1 \; \langle (j_1 \; j_2) \; (j_1 \; j_3) \rangle \,] \in \varpi$, that is $[\, \langle (j_1 \; j_2) \; (j_1 \; j_3) \rangle \;\; j_1] \in \varpi^{-1}$, and shows $\langle j_1 \; j_2 \rangle \in \overline{x_1}^J$ so $[\, \langle (j_1 \; j_2) \; (j_1 \; j_3) \rangle \;\; j_2] \in \varpi^{-1} \overline{x_1}^J$. Furthermore the figure shows $[\, \langle (j_1 \; j_2) \; (j_1 \; j_3) \rangle \;\; \langle (j_1 \; j_2) \; (j_2) \rangle \,] \in \overline{x_1}^K$ and $[\, \langle (j_1 \; j_2) \; (j_2) \rangle \;\; j_2] \in \varpi^{-1}$, so $[\, \langle (j_1 \; j_2) \; (j_1 \; j_3) \rangle \;\; j_2] \in \overline{x_1}^K \varpi^{-1}$ in accordance with the inclusion $\varpi^{-1} \overline{x_1}^J \subseteq \overline{x_1}^K \varpi^{-1}$. Continuing this reasoning confirms $\varpi^{-1} \overline{x_1}^J \subseteq \overline{x_1}^K \varpi^{-1}$, and similarly figure 6.19(b) shows $\varpi^{-1} \overline{x_2}^J \subseteq \overline{x_2}^K \varpi^{-1}$, so ϖ is a one-many weak homomorphism of J to F o R. Indeed ϖ is a one-many weak homomorphism of J "onto" F o R, that is $C[\varpi] = \pi \times \tau$, but this will not be the case in general. Usually the

S_J-covers π and τ satisfying $\pi * \tau = 0(S_J)$ will produce some void intersections, that is for arbitrary $f \in \pi$ and arbitrary $r \in \tau$ perhaps $f \cap r = \emptyset$. Consequently no objective state will be common to these blocks f and r, and the pair $\langle f\ r \rangle \in \pi \times \tau$ will be excluded from the codomain of ω. For the present example π and τ do not produce void intersections, so ω assigns S_J "onto" $\pi \times \tau$.

The preceeding shows that an image F of a semiautomaton J can be used to form a cascade-composite covering of J, by defining a semiautomaton R so that $(f \cap r)\bar{x}^J \neq \emptyset$ implies $\langle r\ r' \rangle \in \overline{\langle x\ f \rangle}^R$ where $(f \cap r)\bar{x}^J \subseteq r'$. To investigate this property assume $x \in X_J$ and assume $\langle j\ j' \rangle \in \bar{x}^J$, as shown in figure 6.20.



$\langle j\ j' \rangle \in \bar{x}^J$ implies $[\langle f\ r \rangle\ \ \langle f'\ r' \rangle] \in \bar{x}^K$

Figure 6.20

Then since π is a S_J-cover there is at least one block $f \in \pi$ where $j \in f$, furthermore $\langle j\ j' \rangle \in \bar{x}^J$ so

$j' \in (f)\overline{x}^J$. Therefore $(f)\overline{x}^J \neq \emptyset$, and F is an image semiautomaton so $(f)\overline{x}^J \neq \emptyset$ implies $\langle f\ f' \rangle \in \overline{x}^F$ for some $f' \in \Pi$ where $(f)\overline{x}^J \subseteq f'$, giving $j' \in f'$. In addition τ is a S_J-cover, so there exists some $r \in \tau$ where $j \in r$, indeed then $f \cap r = \{j\}$ since $\Pi * \tau = 0(S_J)$. Consequently $(f \cap r)\overline{x}^J = \{j'\}$, in which case $(f \cap r)\overline{x}^J \neq \emptyset$ so r is granted a $\overline{\langle x\ f \rangle}^R$-successor r' satisfying $(f \cap r)\overline{x}^J \subseteq r'$. Then $\{j'\} \subseteq r'$, that is $j' \in r'$, and from above $j' \in f'$ where $\Pi * \tau = 0(S_J)$ so $\{j'\} = f' \cap r'$. Finally $\langle f\ f' \rangle \in \overline{x}^F$ and $\langle r\ r' \rangle \in \overline{\langle x\ f \rangle}^R$ implies $[\langle f\ r \rangle\ \langle f'\ r' \rangle] \in \overline{x}^K$, by definition of the composite semiautomaton $K = F \circ R$.

In a sense $\langle f\ f' \rangle \in \overline{x}^F$ is an ambiguous representation of $\langle j\ j' \rangle \in \overline{x}^J$, and occurs since F is an image semiautomaton of J. However the association $\langle r\ r' \rangle \in \overline{\langle x\ f \rangle}^R$ joins with $\langle f\ f' \rangle \in \overline{x}^F$ to resolve the ambiguity, giving a precise simulation $[\langle f\ r \rangle\ \langle f'\ r' \rangle] \in \overline{x}^K$ of the objective association $\langle j\ j' \rangle \in \overline{x}^J$. Hence the association $\langle j\ j' \rangle \in \overline{x}^J$ is "followed" by the association $[\langle f\ r \rangle\ \langle f'\ r' \rangle] \in \overline{x}^K$, and similarly every objective association is followed by a corresponding association in the composite semiautomaton $F \circ R$. This argument can be used as the basis of a formal proof, showing that the natural relation ω from S_J to $\Pi \times \tau$ is a one-many weak homomorphism of J to $F \circ R$.

<u>Theorem</u> [cf: Hartmanis & Stearns; Yoeli]

Let $F = \langle S_F \overline{X}_F \rangle$ be a π-image of a semiautomaton $J = \langle S_J \overline{X}_J \rangle$. Then there exists a semiautomaton $R = \langle S_R \overline{X}_R \rangle$ such that $J \leq F \circ R$.

<u>Proof</u>

Let τ be any S_J-cover such that $\pi * \tau = 0(S_J)$, and associate with each pair $\langle x\ f \rangle \in X_J \times S_F$ a relation $\overline{\langle x\ f \rangle}^{\mathcal{R}}$ over τ where

$$\overline{\langle x\ f \rangle}^{\mathcal{R}} = \left\{ \langle r\ r' \rangle \mid r, r' \in \tau,\ (f \cap r)\overline{x}^J \neq \emptyset \ \&\ (f \cap r)\overline{x}^J \subseteq r' \right\}.$$

Define $R = \langle S_R \overline{X}_R \rangle$ where $S_R = \tau$ and $X_R = X_J \times S_F$, so $\langle x\ f \rangle \in X_J \times S_F$ implies $\overline{\langle x\ f \rangle}^R \in \overline{X}_R$, and let $\overline{\langle x\ f \rangle}^R$ be any mapping derived from the relation $\overline{\langle x\ f \rangle}^{\mathcal{R}}$. Then $\overline{\langle x\ f \rangle}^R \subseteq \overline{\langle x\ f \rangle}^{\mathcal{R}}$, and $D[\overline{\langle x\ f \rangle}^R] = D[\overline{\langle x\ f \rangle}^{\mathcal{R}}]$.

Define $K = \langle S_K \overline{X}_K \rangle$ where $K = F \circ R$, so $S_K = S_F \times S_R$, $X_K = X_F = X_J$, and $x \in X_J$ implies $\overline{x}^K \in \overline{X}_K$ where

$$\overline{x}^K = \left\{ [\langle f\ r \rangle\ \ \langle f'\ r' \rangle] \ \middle|\ \langle f\ f' \rangle \in \overline{x}^F \ \&\ \langle r\ r' \rangle \in \overline{\langle x\ f \rangle}^R \right\}$$

Define the relation

$$\omega = \left\{ [j\ \ \langle f\ r \rangle] \mid j \in S_J,\ \langle f\ r \rangle \in S_F \times S_R,\ j \in f \ \ \&\ \ j \in r \right\}$$

from S_J to $S_F \times S_R$, assume $x \in X_J$ and assume $[\langle f\ r \rangle\ \ j'] \in \omega^{-1}\overline{x}^J$, so $[\langle f\ r \rangle\ \ j] \in \omega^{-1}$ and $\langle j\ j' \rangle \in \overline{x}^J$ for some j. Then $[j\ \ \langle f\ r \rangle] \in \omega$ so $j \in f$ and $j \in r$, in which case $\langle j\ f \rangle \in \pi'$ where π' is the canonical relation from S_J to π. Then $\langle f\ j \rangle \in \pi'^{-1}$, and

$\langle j\ j'\rangle \in \bar{x}^{J}$ so $\langle f\ j'\rangle \in \Pi'^{-1}\bar{x}^{J}$, furthermore Π' is a weak homomorphism of J to Π-image F so $\Pi'^{-1}\bar{x}^{J} \subseteq \bar{x}^{F}\Pi'^{-1}$. Consequently $\langle f\ j'\rangle \in \bar{x}^{F}\Pi'^{-1}$, so $\langle f\ f'\rangle \in \bar{x}^{F}$ and $\langle f'\ j'\rangle \in \Pi'^{-1}$, that is $\langle j'\ f'\rangle \in \Pi'$, for some f'.

From above $j \in f$ and $j \in r$, that is $j \in f \cap r$, in which case $\{j\} = f \cap r$ since $\Pi * \tau = O(S_J)$. Consequently $\{j'\} = (f \cap r)\bar{x}^{J}$, and τ is a S_J-cover so there is at least one $r^{*} \in \tau$ such that $j' \in r^{*}$, in which case $(f \cap r)\bar{x}^{J} \subseteq r^{*}$ so $\langle r\ r^{*}\rangle \in \overline{\langle x\ f\rangle}^{\mathcal{R}}$. This establishes $r \in D[\overline{\langle x\ f\rangle}^{\mathcal{R}}]$, and $D[\overline{\langle x\ f\rangle}^{R}] = D[\overline{\langle x\ f\rangle}^{\mathcal{R}}]$ so $r \in D[\overline{\langle x\ f\rangle}^{R}]$, consequently $\langle r\ r'\rangle \in \overline{\langle x\ f\rangle}^{R}$ for some $r' \in \tau$.

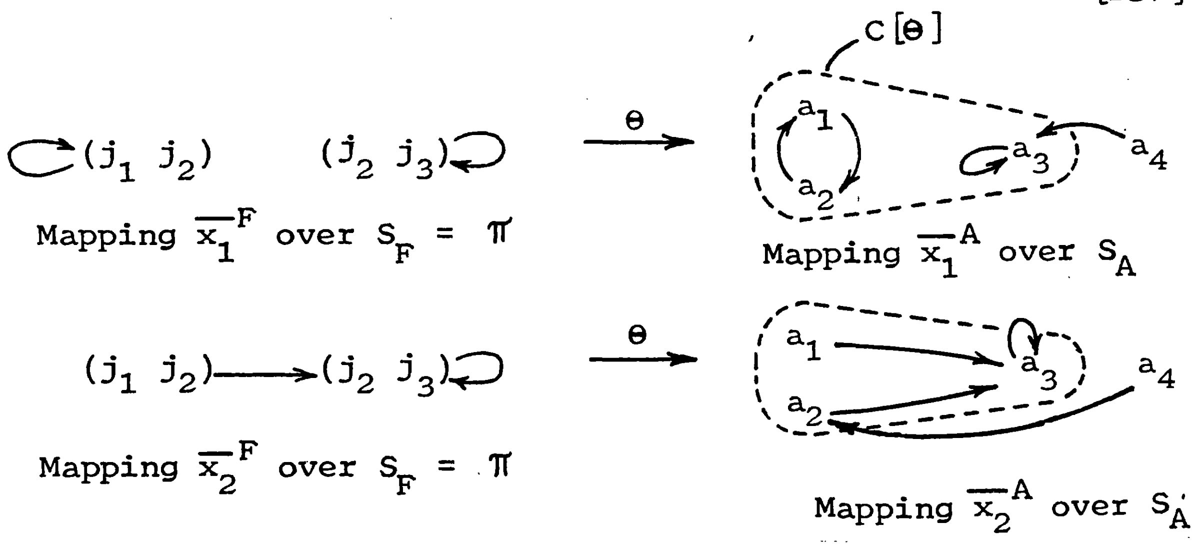
Hence $\langle f\ f'\rangle \in \bar{x}^{F}$ and $\langle r\ r'\rangle \in \overline{\langle x\ f\rangle}^{R}$, so $[\langle f\ r\rangle\ \ \langle f'\ r'\rangle] \in \bar{x}^{K}$. Furthermore $\langle j'\ f'\rangle \in \Pi'$ so $j' \in f'$, and $\langle r\ r'\rangle \in \overline{\langle x\ f\rangle}^{R}$ where $\overline{\langle x\ f\rangle}^{R} \subseteq \overline{\langle x\ f\rangle}^{\mathcal{R}}$ so $(f \cap r)\bar{x}^{J} \subseteq r'$, that is $\{j'\} \subseteq r'$, giving $j' \in r'$. Hence $j' \in f'$ and $j' \in r'$ so $[j'\ \ \langle f'\ r'\rangle] \in \omega$, that is $[\langle f'\ r'\rangle\ \ j'] \in \omega^{-1}$, and $[\langle f\ r\rangle\ \ \langle f'\ r'\rangle] \in \bar{x}^{K}$ so $[\langle f\ r\rangle\ \ j'] \in \bar{x}^{K}\omega^{-1}$.

Hence $[\langle f\ r\rangle\ \ j'] \in \omega^{-1}\bar{x}^{J}$ implies $[\langle f\ r\rangle\ \ j'] \in \bar{x}^{K}\omega^{-1}$, and $x \in X_J$ is arbitrary so $(\forall x)(x \in X_J \implies \omega^{-1}\bar{x}^{J} \subseteq \bar{x}^{K}\omega^{-1})$. Considering now the nature of relation ω, assume $j \in S_J$. Since Π is a S_J-cover there is at least one $f \in \Pi$ such that $j \in f$, similarly τ is a S_J-cover so $j \in r$ for some $r \in \tau$, and then $[j \ \langle f \ r\rangle] \in \omega$. This establishes $D[\omega] = S_J$, and it remains to prove ω to be one-many. Assume $\langle f \ r\rangle \in S_F \times S_R$, and assume $[\langle f \ r\rangle \ j], [\langle f \ r\rangle \ j^*] \in \omega^{-1}$. Then $[j \ \langle f \ r\rangle] \in \omega$ so $j \in f \cap r$, indeed $\{j\} = f \cap r$ since $\pi * \tau = 0(S_J)$. Similarly $[j^* \ \langle f \ r\rangle] \in \omega$ and $f \cap r = \{j^*\}$, so $j^* = j$. Consequently $[\langle f \ r\rangle \ j], [\langle f \ r\rangle \ j^*] \in \omega^{-1}$ implies $j^* = j$, and this shows that ω^{-1} is a mapping. Therefore ω is one-many, and from above ω is a weak homomorphism of J to $K = F \circ R$ so $J \leqslant^{\omega} F \circ R$, completing the proof.

This illustrates the basic approach, and the argument is readily extended to give cascade-composite realisations of a given objective automaton. To appreciate this consider the stock semiautomaton $A = \langle S_A \ \overline{X}_A\rangle$ of figure 6.15, and consider the way this stock semiautomaton relates to the image semiautomaton $F = \langle S_F \ \overline{X}_F\rangle$ of figure 6.17. The relationship is illustrated in figure 6.21, and it is evident that F is related to A by a one-many weak homomorphism θ, that is $F \leqslant^{\theta} A$ where

$$\theta = \{\langle (j_1 \ j_2) \ \ a_1\rangle \ \langle (j_1 \ j_2) \ \ a_2\rangle \ \langle (j_2 \ j_3) \ \ a_3\rangle\}.$$

Consequently, rather than setting out to formalise a semiautomaton R satisfying $J \leqslant F \circ R$, the aim is to define a semiautomaton R so that $J \leqslant A \circ R$.



(a) Image Semiautomaton $F = \langle S_F\ \overline{X}_F \rangle$ (b) Stock Semiautomaton $A = \langle S_A\ \overline{X}_A \rangle$

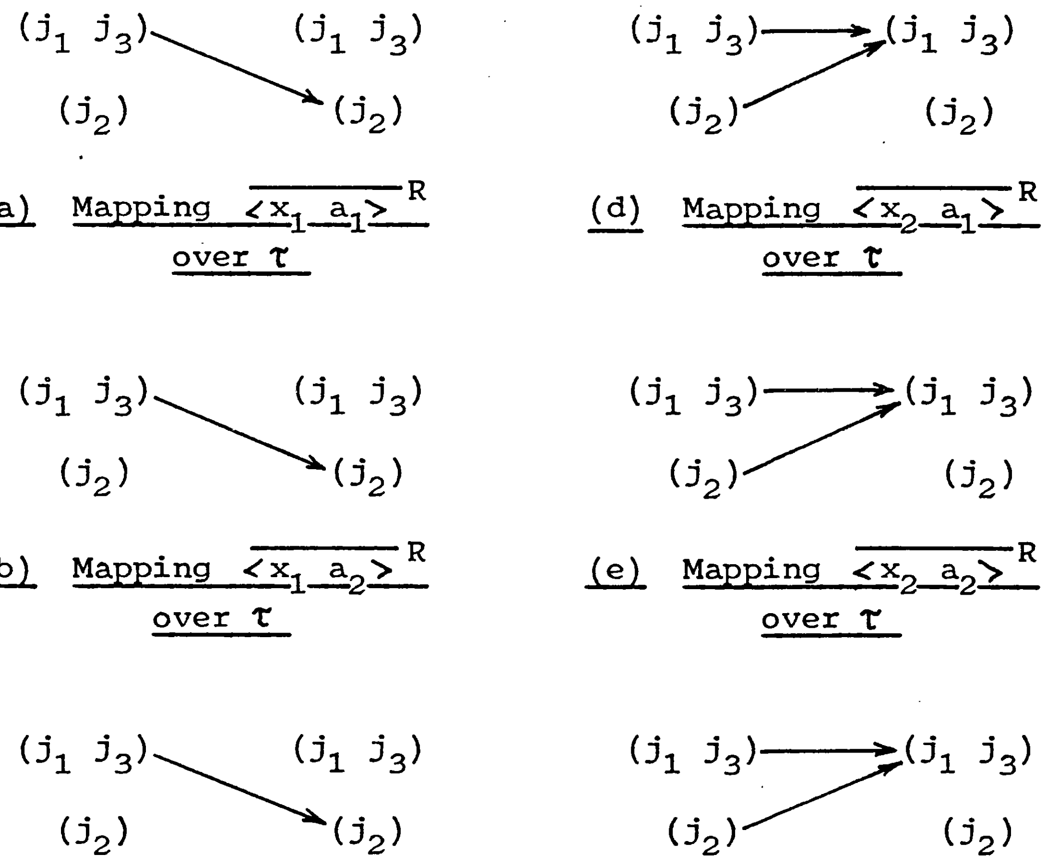
Figure 6.21 $F \leqslant^{\theta} A$

Thus define $R = \langle S_R\ \overline{X}_R \rangle$, where as before $S_R = \tau = (j_1\ j_3)(j_2)$, but define $X_R = X_A \times S_A$ and associate with each pair $\langle x\ a \rangle \in X_A \times S_A$ a mapping $\overline{\langle x\ a \rangle}^R$ over τ, formalised as follows. The relation θ from $S_F = \pi$ to S_A is one-many, so θ^{-1} is a mapping and an arbitrary element $a \in C[\theta]$ will be assigned by θ^{-1} to a particular cover block $f \in \pi$. To find the $\overline{\langle x\ a \rangle}^R$-successor for a given cover block $r \in \tau$, form the intersection $f \cap r$ and consider the set $(f \cap r)\overline{x}^J$. If $(f \cap r)\overline{x}^J = \emptyset$ no $\overline{\langle x\ a \rangle}^R$-successor need be associated with the cover block $r \in \tau$, that is r is excluded from the domain $D[\overline{\langle x\ a \rangle}^R]$. If however $(f \cap r)\overline{x}^J \neq \emptyset$ some $\overline{\langle x\ a \rangle}^R$-successor $r' \in \tau$ for r must be appointed, and r' must satisfy $(f \cap r)\overline{x}^J \subseteq r'$. Since $\pi * \tau = 0(S_J)$ at least one such cover block $r' \in \tau$ will exist, and as before several candidates might exist and an arbitrary candidate can be taken as the $\overline{\langle x\ a \rangle}^R$-successor for r.

Thus for the example $X_A = \{x_1, x_2\}$ and $S_A = \{a_1, a_2, a_3, a_4\}$, so

$$X_A \times S_A = \left\{\begin{matrix} \langle x_1\ a_1\rangle & \langle x_1\ a_2\rangle & \langle x_1\ a_3\rangle & \langle x_1\ a_4\rangle \\ \langle x_2\ a_1\rangle & \langle x_2\ a_2\rangle & \langle x_2\ a_3\rangle & \langle x_2\ a_4\rangle \end{matrix}\right\}$$

and there are eight mappings over τ to be formed. To begin consider the mapping $\overline{\langle x_1\ a_1\rangle}^R$, and determine the block from π to which a_1 is assigned under the mapping θ^{-1}. From above $\langle(j_1\ j_2)\ a_1\rangle \in \theta$ so $\langle a_1\ (j_1\ j_2)\rangle \in \theta^{-1}$, that is θ^{-1} assigns $a_1 \in S_A$ to block $(j_1\ j_2)$ of cover π. Putting $f = (j_1\ j_2)$, the $\overline{\langle x_1\ a_1\rangle}^R$-successor for block $(j_1\ j_3)$ of cover τ can be determined by setting $r = (j_1\ j_3)$. Then $f \cap r = \{j_1\}$, furthermore $\langle j_1\ j_2\rangle \in \overline{x_1}^J$ so $(f \cap r)\overline{x_1}^J = \{j_2\}$. Then since $(f \cap r)\overline{x_1}^J \neq \emptyset$, some $\overline{\langle x_1\ a_1\rangle}^R$-successor r' for r must be appointed and must satisfy $(f \cap r)\overline{x_1}^J \subseteq r'$, that is r' must satisfy $\{j_2\} \subseteq r'$. Clearly $\{j_2\}$ is a subset of just one block of cover τ, this being the block (j_2), so here (j_2) is the only candidate as the $\overline{\langle x_1\ a_1\rangle}^R$-successor for block $(j_1\ j_3)$, that is set $\langle(j_1\ j_3)\ (j_2)\rangle \in \overline{\langle x_1\ a_1\rangle}^R$. Considering now the appointment of a $\overline{\langle x_1\ a_1\rangle}^R$-successor for block (j_2) from cover τ put $r = (j_2)$, in which case $f \cap r = \{j_2\}$. However $j_2 \notin D[\overline{x_1}^J]$ so $(f \cap r)\overline{x_1}^J = \emptyset$, consequently no $\overline{\langle x_1\ a_1\rangle}^R$-successor for block (j_2) of cover τ need be appointed, and mapping $\overline{\langle x_1\ a_1\rangle}^R$ over τ is shown in figure 6.22(a).



(a) Mapping $\overline{\langle x_1\ a_1\rangle}^R$ over τ

(b) Mapping $\overline{\langle x_1\ a_2\rangle}^R$ over τ

(c) Mapping $\overline{\langle x_1\ a_3\rangle}^R$ over τ

(d) Mapping $\overline{\langle x_2\ a_1\rangle}^R$ over τ

(e) Mapping $\overline{\langle x_2\ a_2\rangle}^R$ over τ

(f) Mapping $\overline{\langle x_2\ a_3\rangle}^R$ over τ

Semiautomaton $R = \langle S_R\ \overline{X}_R\rangle$

Figure 6.22

Continuing this reasoning gives the remaining mappings over τ shown in figure 6.22, furthermore a_4 is excluded from the codomain $C[\theta]$ so $\overline{\langle x_1\ a_4\rangle}^R = \emptyset$ and $\overline{\langle x_2\ a_4\rangle}^R = \emptyset$.

It remains to confirm $J \leqslant A \circ R$, and to begin consider the way S_J is related to $S_A \times \tau$. Firstly S_J is related to S_J-cover π by the canonical relation π' from S_J to π, furthermore $F \leqslant^{\theta} A$ so θ relates π to S_A.

Consequently $\Pi'\theta$ is a relation from S_J to S_A, as shown in figure 6.23, and the figure also shows that S_J is related to S_J-cover τ by the canonical relation T from S_J to τ.

$$\begin{tikzcd} & \pi \arrow[r, "\theta"] & S_A \arrow[d, phantom, "\Big\}\ S_A \times \tau" right] \\ S_J \arrow[ur, "\Pi'"] \arrow[rr, "\mathrm{T}"] & & \tau \end{tikzcd}$$

Figure 6.23 Relations Π', T and θ

These relations can be used to express the correspondence between S_J and $S_A \times \tau$ as a relation ϖ, where $[j\ \langle a\ r\rangle] \in \varpi$ iff $\langle j\ a\rangle \in \Pi'\theta$ and $\langle j\ r\rangle \in \mathrm{T}$. That is, $[j\ \langle a\ r\rangle] \in \varpi$ iff $\langle f\ a\rangle \in \theta$ for some cover block f from π where $j \in f$, and r is a block of cover τ so that $j \in r$.

Thus for the present example $\pi = (j_1\ j_2)(j_2\ j_3)$ so

$$\Pi' = \begin{Bmatrix} \langle j_1\ (j_1\ j_2)\rangle & \langle j_3\ (j_2\ j_3)\rangle \\ \langle j_2\ (j_1\ j_2)\rangle & \langle j_2\ (j_2\ j_3)\rangle \end{Bmatrix}$$

furthermore $\tau = (j_1\ j_3)(j_2)$ so

$$\mathrm{T} = \{\langle j_1\ (j_1\ j_3)\rangle\ \langle j_2\ (j_2)\rangle\ \langle j_3\ (j_1\ j_3)\rangle\},$$

and from above

$$\theta = \{\langle (j_1\ j_2)\ a_1\rangle\ \langle (j_1\ j_2)\ a_2\rangle\ \langle (j_2\ j_3)\ a_3\rangle\}.$$

Consequently

$$\Pi'\theta = \begin{Bmatrix} \langle j_1\ a_1\rangle & \langle j_2\ a_1\rangle & \langle j_2\ a_3\rangle \\ \langle j_1\ a_2\rangle & \langle j_2\ a_2\rangle & \langle j_3\ a_3\rangle \end{Bmatrix}$$

and by definition

$$\omega = \left\{ [j \ \langle a \ r \rangle] \,\middle|\, \langle j \ a \rangle \in \Pi\!\!\!/ \theta \ \& \ \langle j \ r \rangle \in \mathcal{T} \right\} \text{ so}$$

$$\omega = \left\{ \begin{matrix} [j_1 & \langle a_1 \ (j_1 \ j_3) \rangle] & [j_2 & \langle a_2 \ (j_2) \rangle] \\ [j_1 & \langle a_2 \ (j_1 \ j_3) \rangle] & [j_2 & \langle a_3 \ (j_2) \rangle] \\ [j_2 & \langle a_1 \ (j_2) \rangle] & [j_3 & \langle a_3 \ (j_1 \ j_3)] \end{matrix} \right\}$$

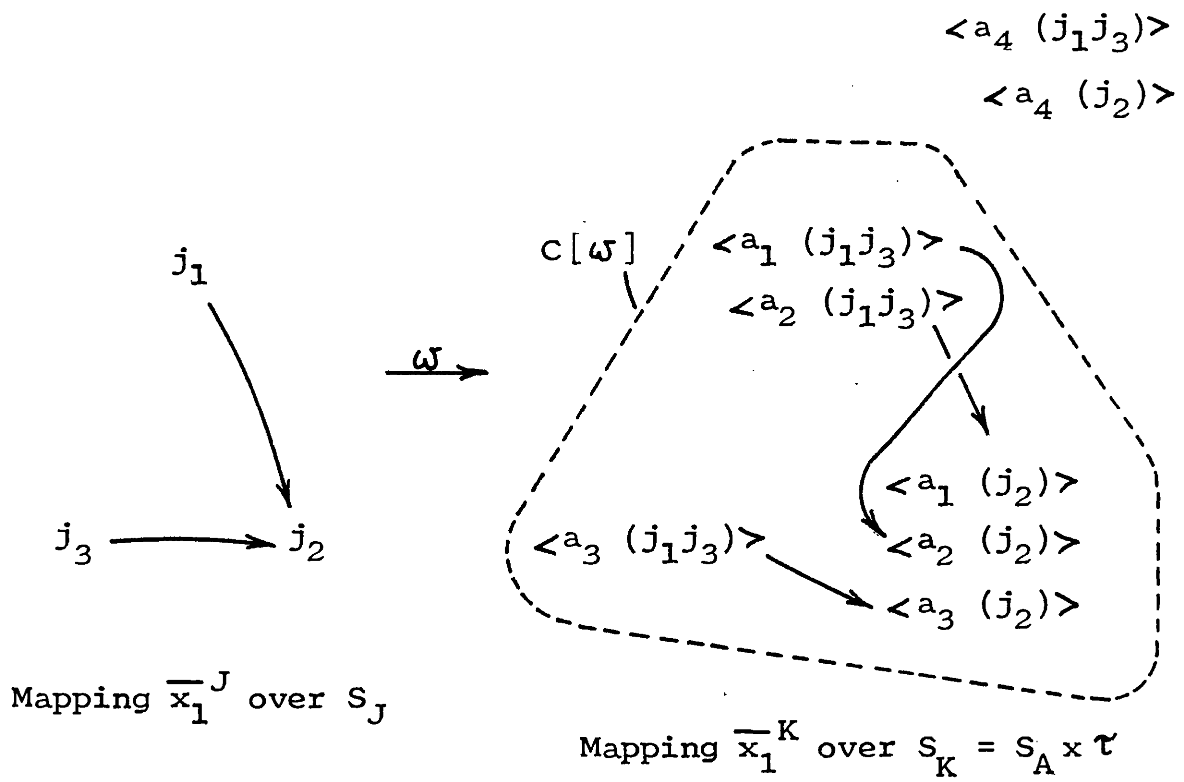
Clearly ω is a one-many relation from S_J to $S_A \times \tau$, so ω relates semiautomaton J to the cascade-composite semiautomaton $A \circ R$. Define $K = \langle S_K \ \overline{X}_K \rangle$ where $K = A \circ R$, so $S_K = S_A \times \tau$, $X_K = X_A = \{x_1, x_2\}$ and $x \in X_K$ implies $\overline{x}^K \in \overline{X}_K$ where

$$\overline{x}^K = \left\{ [\langle a \ r \rangle \ \ \langle a' \ r' \rangle] \,\middle|\, \langle a \ a' \rangle \in \overline{x}^A \ \& \ \langle r \ r' \rangle \in \overline{\langle x \ a \rangle}^R \right\}.$$

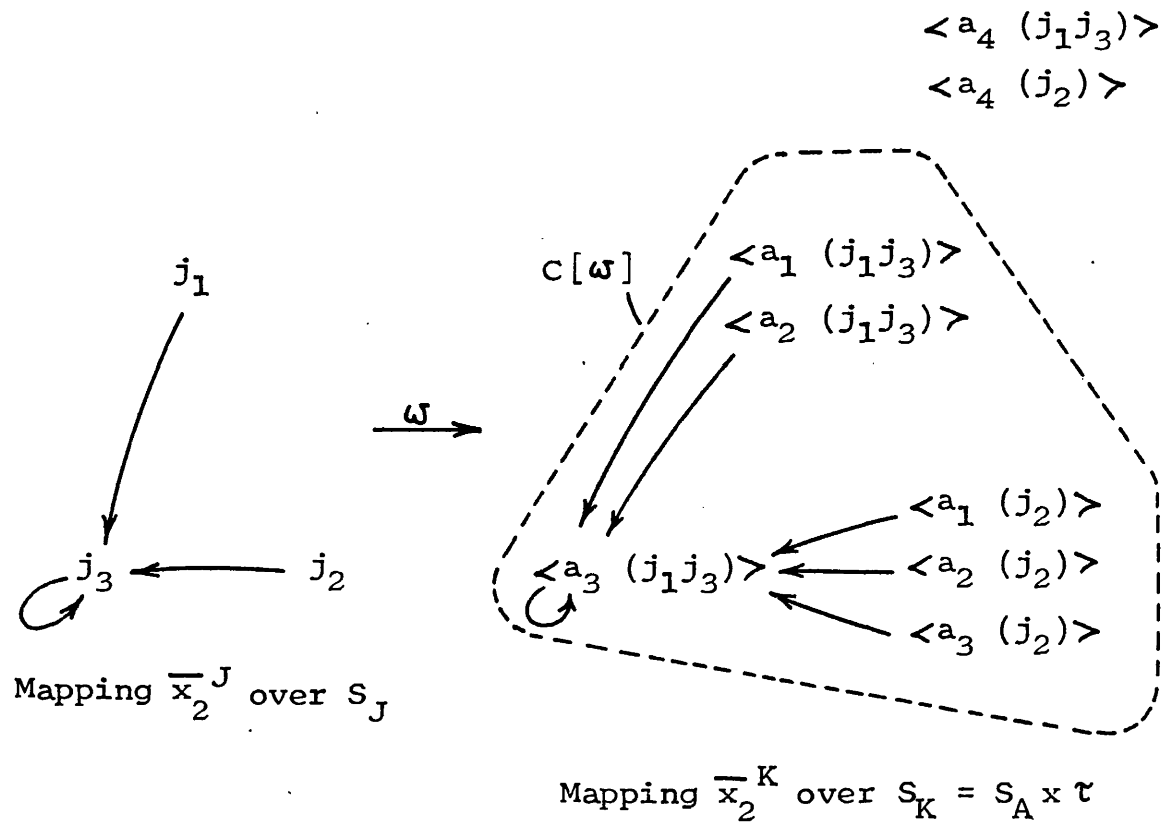
For example $[\langle a_1 \ (j_1 \ j_3) \rangle \ \ \langle a_2 \ (j_2) \rangle] \in \overline{x_1}^K$, since figure 6.21 shows $\langle a_1 \ a_2 \rangle \in \overline{x_1}^A$ and figure 6.22 shows $\langle (j_1 \ j_3) \ (j_2) \rangle \in \overline{\langle x_1 \ a_1 \rangle}^R$. Continuing this reasoning gives the mappings $\overline{x_1}^K$ and $\overline{x_2}^K$ over $S_A \times \tau$, as expressed in the table.

	x_1	x_2
$\langle a_1 \ (j_1 \ j_3) \rangle$	$\langle a_2 \ (j_2) \rangle$	$\langle a_3 \ (j_1 \ j_3) \rangle$
$\langle a_2 \ (j_1 \ j_3) \rangle$	$\langle a_1 \ (j_2) \rangle$	$\langle a_3 \ (j_1 \ j_3) \rangle$
$\langle a_3 \ (j_1 \ j_3) \rangle$	$\langle a_3 \ (j_2) \rangle$	$\langle a_3 \ (j_1 \ j_3) \rangle$
$\langle a_4 \ (j_1 \ j_3) \rangle$	-	-
$\langle a_1 \ (j_2) \rangle$	-	$\langle a_3 \ (j_1 \ j_3) \rangle$
$\langle a_2 \ (j_2) \rangle$	-	$\langle a_3 \ (j_1 \ j_3) \rangle$
$\langle a_3 \ (j_2) \rangle$	-	$\langle a_3 \ (j_1 \ j_3) \rangle$
$\langle a_4 \ (j_2) \rangle$	-	-

Semiautomaton $K = \langle S_K \ \overline{X}_K \rangle$, where $K = A \circ R$



(a) Showing $\omega^{-1}\overline{x}_1^{\,J} \subseteq \overline{x}_1^{\,K}\omega^{-1}$



(b) Showing $\omega^{-1}\overline{x}_2^{\,J} \subseteq \overline{x}_2^{\,K}\omega^{-1}$

Figure 6.24 $J \leqslant^{\omega} A \circ R$

Then the relation ω from S_J to $S_A \times \tau$ is a one-many weak homomorphism of J to K = A o R, and this is shown in figure 6.24, where the relation ω is expressed implicitly and the mappings $\overline{x_1}^K, \overline{x_2}^K$ over $S_A \times \tau$ are those expressed by the table defining semiautomaton K = A o R. For example figure 6.24(a) shows $[j_1 \ \langle a_1 \ (j_1 \ j_3)\rangle\ ] \in \omega$, that is $[\langle a_1 \ (j_1 \ j_3)\rangle \ \ j_1] \in \omega^{-1}$, and shows $\langle j_1 \ j_2\rangle \in \overline{x_1}^J$ so $[\langle a_1 \ (j_1 \ j_3)\rangle \ \ j_2] \in \omega^{-1}\overline{x_1}^J$. Furthermore $[\langle a_1 \ (j_1 \ j_3)\rangle \ \ j_2] \in \overline{x_1}^K \omega^{-1}$, in accordance with the inclusion $\omega^{-1}\,\overline{x_1}^J \subseteq \overline{x_1}^K \omega^{-1}$, since figure 6.24(a) shows $[\langle a_1 \ (j_1 \ j_3)\rangle \ \ \langle a_2 \ (j_2)\rangle\ ] \in \overline{x_1}^K$ and shows $[\langle a_2 \ (j_2)\rangle \ \ j_2] \in \omega^{-1}$. Similarly figure 6.24(b) shows $\omega^{-1}\overline{x_2}^J \subseteq \overline{x_2}^K \omega^{-1}$, and ω is one-many with S_J as domain so $J \leqslant^{\omega} A \circ R$.

Theorem

If F is an image semiautomaton of a semiautomaton J, and A is a semiautomaton where $F \leqslant A$, then there exists a semiautomaton R so that $J \leqslant A \circ R$.

Proof

Define $J = \langle S_J \ \overline{X}_J\rangle$, let $F = \langle S_F \ \overline{X}_F\rangle$ be a π-image of J, and assume $F \leqslant^{\theta} A$ where $A = \langle S_A \ \overline{X}_A\rangle$. Let τ be any S_J-cover such that $\pi * \tau = 0(S_J)$, and associate with each pair $\langle x \ a\rangle \in X_A \times S_A$ a relation $\overline{\langle x \ a\rangle}^{\mathcal{R}}$ over τ where, for π' the canonical relation associated with π,

$$\overline{\langle x \ a\rangle}^{\mathcal{R}} = \left\{ \langle r \ r'\rangle \;\middle|\; \begin{array}{l} \langle r \ r'\rangle \in \tau \times \tau, ([a]\theta^{-1}\pi'^{-1} \cap r)\overline{x}^J \neq \emptyset, \\ \& \quad ([a]\theta^{-1}\pi'^{-1} \cap r)\,\overline{x}^J \subseteq r' \end{array} \right\}$$

Define $R = \langle S_R\ \overline{X}_R \rangle$ where $S_R = \tau$ and $X_R = X_A \times S_A$, so $\langle x\ a\rangle \in X_A \times S_A$ implies $\overline{\langle x\ a\rangle}^R \in \overline{X}_R$, and let $\overline{\langle x\ a\rangle}^R$ be an arbitrary mapping derived from the relation $\overline{\langle x\ a\rangle}^{\mathcal{R}}$ over τ, so $\overline{\langle x\ a\rangle}^R \subseteq \overline{\langle x\ a\rangle}^{\mathcal{R}}$ and $D[\overline{\langle x\ a\rangle}^R] = D[\overline{\langle x\ a\rangle}^{\mathcal{R}}]$.

Define $K = \langle S_K\ \overline{X}_K \rangle$ where $K = A \circ R$, so $S_K = S_A \times S_R$, $X_K = X_A$, and $x \in X_A$ implies $\overline{x}^K \in \overline{X}_K$ where $\overline{x}^K = \{[\langle a\ r\rangle\ \ \langle a'\ r'\rangle] \mid \langle a\ a'\rangle \in \overline{x}^A \ \&\ \langle r\ r'\rangle \in \overline{\langle x\ a\rangle}^R\}$.
Define the relation ϖ from S_J to $S_A \times S_R$ where, for Π' the canonical relation from S_J to S_J-cover π and T the canonical relation from S_J to S_J-cover τ,

$$\varpi = \left\{ [j\ \ \langle a\ r\rangle] \;\middle|\; \begin{array}{l} j \in S_J,\ \langle a\ r\rangle \in S_A \times S_R, \\ \langle j\ a\rangle \in \Pi'\theta \ \ \&\ \ \langle j\ r\rangle \in \mathrm{T} \end{array} \right\}$$

Assume $x \in X_J$, in which case $x \in X_F$ and $x \in X_A$ since $X_F = X_J = X_A$, and assume $[\langle a\ r\rangle\ \ j'] \in \varpi^{-1}\overline{x}^J$. Then $[\langle a\ r\rangle\ \ j] \in \varpi^{-1}$ and $\langle j\ j'\rangle \in \overline{x}^J$, for some j, and then $[j\ \ \langle a\ r\rangle] \in \varpi$ so $\langle j\ a\rangle \in \Pi'\theta$ and $\langle j\ r\rangle \in \mathrm{T}$. Therefore $\langle j\ f\rangle \in \Pi'$ and $\langle f\ a\rangle \in \theta$ for some f, in which case $\langle f\ j\rangle \in \Pi'^{-1}$, and $\langle j\ j'\rangle \in \overline{x}^J$ so $\langle f\ j'\rangle \in \Pi'^{-1}\overline{x}^J$. Furthermore Π' is a weak homomorphism of J to image semiautomaton F so $\Pi'^{-1}\overline{x}^J \subseteq \overline{x}^F\Pi'^{-1}$, consequently $\langle f\ j'\rangle \in \Pi'^{-1}\overline{x}^J$ implies $\langle f\ j'\rangle \in \overline{x}^F\,\Pi'^{-1}$. Therefore $\langle f\ f'\rangle \in \overline{x}^F$ and $\langle f'\ j'\rangle \in \Pi'^{-1}$ for some f', and from above $\langle f\ a\rangle \in \theta$, that is $\langle a\ f\rangle \in \theta^{-1}$, and $\langle f\ f'\rangle \in \overline{x}^F$ so $\langle a\ f'\rangle \in \theta^{-1}\overline{x}^F$. Furthermore $F \leq^{\theta} A$ so $\theta^{-1}\overline{x}^F \subseteq \overline{x}^A\theta^{-1}$, therefore $\langle a\ f'\rangle \in \theta^{-1}\overline{x}^F$

implies $\langle a\ f'\rangle \in \bar{x}^A\theta^{-1}$, in which case $\langle a\ a'\rangle \in \bar{x}^A$ and $\langle a'\ f'\rangle \in \theta^{-1}$, that is $\langle f'\ a'\rangle \in \theta$, for some a'. Then $\langle j'\ a'\rangle \in \Pi'\theta$, since $\langle j'\ f'\rangle \in \Pi'$ and $\langle f'\ a'\rangle \in \theta$.

Furthermore $\langle j\ a\rangle \in \Pi'\theta$ implies $j \in [a](\Pi'\theta)^{-1}$, that is $j \in [a]\theta^{-1}\Pi'^{-1}$, and from above $\langle j\ r\rangle \in T$ so $j \in r$. Consequently $j \in [a]\theta^{-1}\Pi'^{-1} \cap r$, in which case $\{j\} = [a]\theta^{-1}\Pi'^{-1} \cap r$ since $\Pi * \tau = O(S_J)$, and then $\{j'\} = ([a]\theta^{-1}\Pi'^{-1} \cap r)\bar{x}^J$ since $\langle j\ j'\rangle \in \bar{x}^J$. Clearly $([a]\theta^{-1}\Pi'^{-1} \cap r)\bar{x}^J \neq \emptyset$, furthermore τ is a S_J-cover so $j' \in r^*$ for some $r^* \in \tau$, giving $\langle r\ r^*\rangle \in \overline{\langle x\ a\rangle}^{\mathcal{R}}$. Then $r \in D[\overline{\langle x\ a\rangle}^{\mathcal{R}}]$, and $D[\overline{\langle x\ a\rangle}^{R}] = D[\overline{\langle x\ a\rangle}^{\mathcal{R}}]$ so $r \in D[\overline{\langle x\ a\rangle}^{R}]$. Hence $\langle r\ r'\rangle \in \overline{\langle x\ a\rangle}^{R}$ for some $r' \in \tau$, and $\overline{\langle x\ a\rangle}^{R} \subseteq \overline{\langle x\ a\rangle}^{\mathcal{R}}$ so $([a]\theta^{-1}\Pi'^{-1} \cap r)\bar{x}^J \subseteq r'$, that is $\{j'\} \subseteq r'$, so $\langle j'\ r'\rangle \in T$.

Consequently $\langle j'\ r'\rangle \in T$ and $\langle j'\ a'\rangle \in \Pi'\theta$, in which case $[j'\ \langle a'\ r'\rangle] \in \omega$, furthermore $\langle a\ a'\rangle \in \bar{x}^A$ and $\langle r\ r'\rangle \in \overline{\langle x\ a\rangle}^{R}$ so $[\langle a\ r\rangle\ \langle a'\ r'\rangle] \in \bar{x}^K$. Hence $[\langle a\ r\rangle\ \langle a'\ r'\rangle] \in \bar{x}^K$ and $[\langle a'\ r'\rangle\ j'] \in \omega^{-1}$, so $[\langle a\ r\rangle\ j'] \in \bar{x}^K\omega^{-1}$.

Therefore $[\langle a\ r\rangle\ j'] \in \omega^{-1}\bar{x}^J$ implies $[\langle a\ r\rangle\ j'] \in \bar{x}^K\omega^{-1}$, and $x \in X_J$ is arbitrary so $(\forall x)(x \in X_J \Longrightarrow \omega^{-1}\bar{x}^J \subseteq \bar{x}^K\omega^{-1})$. Considering now the nature of relation ω clearly $D[\omega] \subseteq S_J$, so assume $j \in S_J$. Since τ is a S_J-cover $\langle j\ r\rangle \in T$ for at least one $r \in \tau$, imilarly Π is a S_J-cover so $\langle j\ f\rangle \in \Pi'$ for at least one $f \in \Pi$. Furthermore $D[\theta] = \Pi$, since $F \leqslant^{\theta} A$ and $\Pi = S_F$, so $\langle f\ a\rangle \in \theta$ for some $a \in S_A$. Consequently $\langle j\ a\rangle \in \Pi/\theta$, and $\langle j\ r\rangle \in \mathrm{T}$ so $[j\ \langle a\ r\rangle] \in \omega$. Therefore $j \in S_J$ implies $j \in D[\omega]$, so $S_J \subseteq D[\omega]$, and from above $D[\omega] \subseteq S_J$ so $D[\omega] = S_J$.

This confirms ω to be a weak homomorphism of J to $A \circ R$, and it remains to prove ω to be a one-many. Assume $[\langle a\ r\rangle\ j], [\langle a\ r\rangle\ j^*] \in \omega^{-1}$. Then $[j\ \langle a\ r\rangle] \in \omega$, in which case $\langle j\ a\rangle \in \Pi/\theta$ and $\langle j\ r\rangle \in \mathrm{T}$, furthermore $\langle j\ a\rangle \in \Pi/\theta$ implies $\langle j\ f\rangle \in \Pi$ and $\langle f\ a\rangle \in \theta$, for some f. Hence $\langle j\ f\rangle \in \Pi$ and $\langle j\ r\rangle \in \mathrm{T}$, that is $j \in f$ and $j \in r$, so $j \in f \cap r$ and then $\{j\} = f \cap r$ since $\Pi * \mathrm{T} = O(S_J)$. Similarly $[j^*\ \langle a\ r\rangle] \in \omega$ implies $\langle j^*\ a\rangle \in \Pi/\theta$ and $\langle j^*\ r\rangle \in \mathrm{T}$, so $\langle j^*\ f^*\rangle \in \Pi$ and $\langle f^*\ a\rangle \in \theta$ for some f^*. Hence $j^* \in f^*$ and $j^* \in r$, that is $j^* \in f^* \cap r$, and $\Pi * \mathrm{T} = O(S_J)$ so $\{j^*\} = f^* \cap r$. However from above $\langle f\ a\rangle \in \theta$ and $\langle f^*\ a\rangle \in \theta$, that is $\langle a\ f\rangle \in \theta^{-1}$ and $\langle a\ f^*\rangle \in \theta^{-1}$, and θ^{-1} is a mapping since θ is one-many, therefore $f = f^*$. Consequently $\{j^*\} = f^* \cap r$ becomes $\{j^*\} = f \cap r$, and from above $\{j\} = f \cap r$ so $j = j^*$.

Hence $[\langle a\ r\rangle\ j], [\langle a\ r\rangle\ j^*] \in \omega^{-1}$ implies $j = j^*$, so ω^{-1} is a mapping, that is ω is one-many. Therefore ω is a one-many weak homomorphism of J to $K = A \circ R$, that is $J \leqslant^{\omega} A \circ R$, completing the proof.

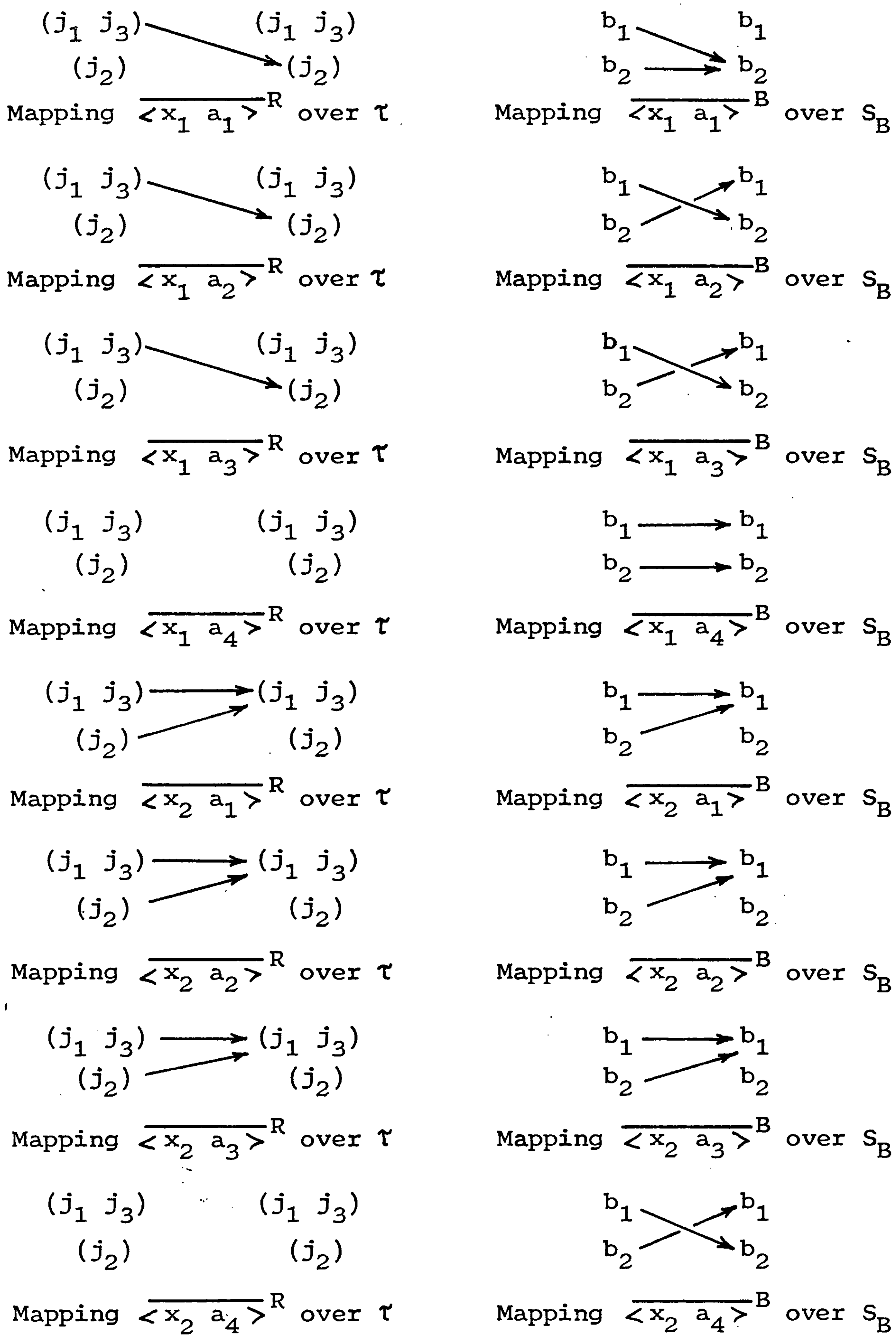
The theorem is directly related to the cascade realisation of a given automaton $\hat{J}$. The first step is to find a stock semiautomaton A covering an image semiautomaton F of the semiautomaton J, and then the theorem ensures that an automaton R can be formalised so that $J \leqslant A \circ R$. The final step is to find a realisation for the semiautomaton R. For example semiautomaton R of figure 6.22 is closely related to the stock semiautomaton B of figure 6.15, and this is illustrated in figure 6.25. The correspondence can be formalised as the relation

$$\mathfrak{S} = \{ \langle (j_1 \; j_3) \quad b_1 \rangle \; \langle (j_2) \quad b_2 \rangle \}$$

from S_R to S_B, and then $\mathfrak{S}$ is a one-many weak homomorphism of R to B, that is $R \leqslant^{\mathfrak{S}} B$. In fact $\mathfrak{S}$ is bijective, and is therefore a partial isomorphism of R to B, but this is unimportant and the essential feature is that the weak homomorphism $\mathfrak{S}$ is one-many.

Then $S_A \times S_R$ is naturally related to $S_A \times S_B$, for example let $\langle a \; r \rangle$ be an arbitrary element from $S_A \times S_R$, in which case $r \in S_R$. Since $\mathfrak{S}$ has S_R as domain there must be at least one $b \in S_B$ where $\langle r \; b \rangle \in \mathfrak{S}$, consequently the arbitrary pair $\langle a \; r \rangle \in S_A \times S_R$ is associated with at least one pair $\langle a \; b \rangle \in S_A \times S_B$ where $\langle r \; b \rangle \in \mathfrak{S}$. This can be formalised as a one-many relation σ from $S_A \times S_R$ to $S_A \times S_B$, where

$$\sigma = \left\{ [\langle a \; r \rangle \; \langle a \; b \rangle] \;\middle|\; \begin{array}{c} \langle a \; r \rangle \in S_A \times S_R, \langle a \; b \rangle \in S_A \times S_B \\ \& \; \langle r \; b \rangle \in \mathfrak{S} \end{array} \right\}$$



(a) Semiautomaton $R = \langle S_R\ \overline{X}_R\rangle$

(b) Semiautomaton $B = \langle S_B\ \overline{X}_B\rangle$

Figure 6.25 $R \leqslant^S B$

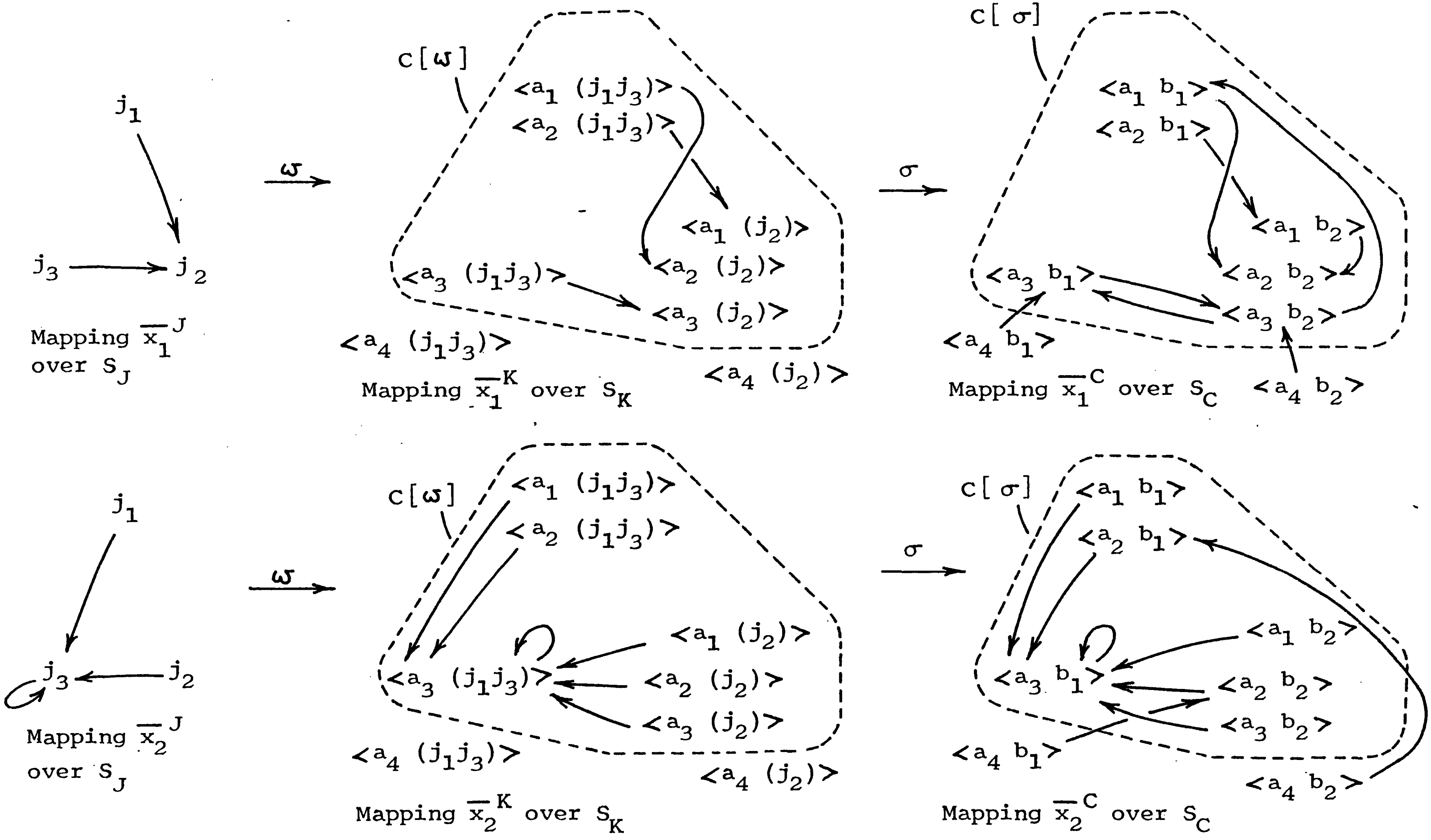
Thus for the present example

$$S_A \times S_R = \left\{\begin{array}{ll} \langle a_1\ (j_1\ j_3)\rangle & \langle a_1\ (j_2)\rangle \\ \langle a_2\ (j_1\ j_3)\rangle & \langle a_2\ (j_2)\rangle \\ \langle a_3\ (j_1\ j_3)\rangle & \langle a_3\ (j_2)\rangle \\ \langle a_4\ (j_1\ j_3)\rangle & \langle a_4\ (j_2)\rangle \end{array}\right\}$$

and $\mathcal{S} = \{\langle (j_1\ j_3)\ b_1\rangle\ \langle (j_2)\ b_2\rangle\}$ so

$$\sigma = \left\{\begin{array}{l} [\langle a_1\ (j_1\ j_3)\rangle\ \ \langle a_1\ b_1\rangle][\langle a_1\ (j_2)\rangle\ \ \langle a_1\ b_2\rangle] \\ [\langle a_2\ (j_1\ j_3)\rangle\ \ \langle a_2\ b_1\rangle][\langle a_2\ (j_2)\rangle\ \ \langle a_2\ b_2\rangle] \\ [\langle a_3\ (j_1\ j_3)\rangle\ \ \langle a_3\ b_1\rangle][\langle a_3\ (j_2)\rangle\ \ \langle a_3\ b_2\rangle] \\ [\langle a_4\ (j_1\ j_3)\rangle\ \ \langle a_4\ b_1\rangle][\langle a_4\ (j_2)\rangle\ \ \langle a_4\ b_2\rangle] \end{array}\right\}.$$

Here the relation σ from $S_A \times S_R$ to $S_A \times S_B$ is injective, rather than one-many, and it is also evident that σ establishes a correspondence between semiautomaton $A \circ R$ and semiautomaton $A \circ B$. To investigate this observe that semiautomaton $C = A \circ B$ has been defined in figure 6.15, and that the mappings $\overline{x_1}^K$ and $\overline{x_2}^K$ associated with semiautomaton $K = A \circ R$ have been given in figure 6.24. Then the above relation σ is a one-many weak homomorphism of $K = A \circ R$ to $C = A \circ B$, and this is shown in figure 6.26, where σ is expressed implicitly in the normal way. For example figure 6.26(a) shows $[\langle a_1\ (j_1\ j_3)\rangle\ \langle a_1\ b_1\rangle] \in \sigma$, that is $[\langle a_1\ b_1\rangle\ \langle a_1\ (j_1\ j_3)\rangle] \in \sigma^{-1}$, and shows $[\langle a_1\ (j_1\ j_3)\rangle\ \ \langle a_2\ (j_2)\rangle] \in \overline{x_1}^K$ so $[\langle a_1\ b_1\rangle\ \ \langle a_2\ (j_2)\rangle] \in \sigma^{-1}\overline{x_1}^K$. Furthermore $[\langle a_1\ b_1\rangle\ \ \langle a_2\ (j_2)\rangle] \in \overline{x_1}^C\sigma^{-1}$, in accordance with the inclusion $\sigma^{-1}\overline{x_1}^K \subseteq \overline{x_1}^C\sigma^{-1}$, since figure 6.26 shows $[\langle a_1\ b_1\rangle\ \ \langle a_2\ b_2\rangle] \in \overline{x_1}^C$ and $[\langle a_2\ b_2\rangle\ \ \langle a_2\ (j_2)\rangle] \in \sigma^{-1}$.



<u>Figure 6.26</u> <u>$J \leqslant^{\omega} K \leqslant^{\sigma} C$</u>

Hence by inspection $\sigma^{-1}\overline{x}_1^{\,K} \subseteq \overline{x}_1^{\,C}\sigma^{-1}$, and similarly figure 6.26 shows $\sigma^{-1}\overline{x}_2^{\,K} \subseteq \overline{x}_2^{\,C}\sigma^{-1}$ so $K \leqslant^{\sigma} C$, that is $A \circ R \leqslant^{\sigma} A \circ B$. However from previously $J \leqslant^{\omega} A \circ R$, giving $J \leqslant^{\omega} A \circ R \leqslant^{\sigma} A \circ B$, and this is illustrated in figure 6.26 by reproducing figure 6.24. For example figure 6.26 shows that the association $[\langle a_1\ b_1\rangle\ \ \langle a_2\ b_2\rangle\,] \in \overline{x}_1^{\,C}$ represents the association
$[\langle a_1\ (j_1\ j_3)\rangle\ \langle a_2\ (j_2)\rangle] \in \overline{x}_1^{\,K}$, which in turn represents the association $\langle j_1\ j_2\rangle \in \overline{x}_1^{\,J}$. Clearly $\omega\sigma$ is a one-many weak homorphism of J to $A \circ B$, that is $J \leqslant^{\omega\sigma} A \circ B$, so the stock units A and B can be used to form a cascade realisation of the objective automaton. Furthermore $\gamma = \omega\sigma$, so figure 6.26 shows detail omitted from figure 6.16, where $J \leqslant^{\gamma} A \circ B$.

<u>Theorem</u>

If $J \leqslant A \circ R$ and $R \leqslant B$, then $J \leqslant A \circ B$.

<u>Proof</u>

Assume $J \leqslant^{\omega} A \circ R$, and assume $R \leqslant^{\delta} B$. Define $K = \langle S_K\ \overline{X}_K\rangle$ where $K = A \circ R$, so $S_K = S_A \times S_R$, $X_K = X_A$, and $x \in X_K$ implies $\overline{x}^K \in \overline{X}_K$ where

$$\overline{x}^K = \left\{[\langle a\ r\rangle\ \ \langle a'\ r'\rangle\,]\ \middle|\ \langle a\ a'\rangle \in \overline{x}^A\ \&\ \langle r\ r'\rangle \in \overline{\langle x\ a\rangle}^{\,R}\right\}.$$

Define $C = \langle S_C\ \overline{X}_C\rangle$ where $C = A \circ B$, so $S_C = S_A \times S_B$, $X_C = X_A$, and $x \in X_C$ implies $\overline{x}^C \in \overline{X}_C$ where

$$\overline{x}^C = \left\{[\langle a\ b\rangle\ \ \langle a'\ b'\rangle]\ \middle|\ \langle a\ a'\rangle \in \overline{x}^A\ \&\ \langle b\ b'\rangle \in \overline{\langle x\ a\rangle}^{\,B}\right\}.$$

Assuming $R = \langle S_R\ \overline{X}_R\rangle$ and $B = \langle S_B\ \overline{X}_B\rangle$, relation δ from S_R to S_B is one-many since $R \leqslant^{\delta} B$, and induces a relation σ from $S_A \times S_R$ to $S_A \times S_B$ where

$$\sigma = \left\{ [\langle a\ r\rangle \quad \langle a\ b\rangle] \;\middle|\; \begin{array}{l} \langle a\ r\rangle \in S_A \times S_R,\ \langle a\ b\rangle \in S_A \times S_B \\ \qquad \& \quad \langle r\ b\rangle \in \varsigma \end{array} \right\}$$

Assume $x \in X_K$, so $x \in X_C$ since $X_K = X_A = X_C$, and assume $[\langle a\ b\rangle \quad \langle a'\ r'\rangle] \in \sigma^{-1}\bar{x}^K$, so $[\langle a\ b\rangle \quad \langle a\ r\rangle] \in \sigma^{-1}$ and $[\langle a\ r\rangle \quad \langle a'\ r'\rangle] \in \bar{x}^K$ for some $\langle a\ r\rangle$. Clearly $[\langle a\ r\rangle \quad a] \in \emptyset_A$, that is $[a \quad \langle a\ r\rangle] \in \emptyset_A{}^{-1}$, where $\emptyset_A$ is the projection of $S_A \times S_R$ onto S_A, and $[\langle a\ r\rangle \quad \langle a'\ r'\rangle] \in \bar{x}^K$ so $[a \quad \langle a'\ r'\rangle] \in \emptyset_A{}^{-1}\bar{x}^K$. Furthermore, projection $\emptyset_A$ is a weak homomorphism (indeed a partial epimorphism) of $K = A \circ R$ onto semiautomaton A, so $\emptyset_A{}^{-1}\bar{x}^K \subseteq \bar{x}^A\emptyset_A{}^{-1}$, therefore $[a \quad \langle a'\ r'\rangle] \in \emptyset_A{}^{-1}\bar{x}^K$ implies $[a \quad \langle a'\ r'\rangle] \in \bar{x}^A\emptyset_A{}^{-1}$. Consequently $\langle a\ \alpha\rangle \in \bar{x}^A$ and $[\alpha \quad \langle a'\ r'\rangle] \in \emptyset_A{}^{-1}$ for some α, but then $[\langle a'\ r'\rangle \quad \alpha] \in \emptyset_A$ so $\alpha = a'$, giving $\langle a\ a'\rangle \in \bar{x}^A$ and $[\langle a'\ r'\rangle \quad a'] \in \emptyset_A$.

From above $[\langle a\ b\rangle \quad \langle a\ r\rangle] \in \sigma^{-1}$, that is $[\langle a\ r\rangle \quad \langle a\ b\rangle] \in \sigma$, so $\langle r\ b\rangle \in \varsigma$. Therefore $\langle b\ r\rangle \in \varsigma^{-1}$, in addition $[\langle a\ r\rangle \quad \langle a'\ r'\rangle] \in \bar{x}^K$ so $\langle r\ r'\rangle \in \overline{\langle x\ a\rangle}^R$. Consequently $\langle b\ r'\rangle \in \varsigma^{-1}\overline{\langle x\ a\rangle}^R$, and $R \leqslant^{\varsigma} B$ so $\varsigma^{-1}\overline{\langle x\ a\rangle}^R \subseteq \overline{\langle x\ a\rangle}^B\varsigma^{-1}$, giving $\langle b\ r'\rangle \in \overline{\langle x\ a\rangle}^B\varsigma^{-1}$. Therefore $\langle b\ b'\rangle \in \overline{\langle x\ a\rangle}^B$ and $\langle b'\ r'\rangle \in \varsigma^{-1}$, that is $\langle r'\ b'\rangle \in \varsigma$, for some b'.

Hence $\langle a\ a'\rangle \in \bar{x}^A$ and $\langle b\ b'\rangle \in \overline{\langle x\ a\rangle}^B$, so $[\langle a\ b\rangle \quad \langle a'\ b'\rangle] \in \bar{x}^C$, in addition

$\langle a' \; r' \rangle \in S_A \times S_R$, $\langle a' \; b' \rangle \in S_A \times S_B$ and
$\langle r' \; b' \rangle \in \mathcal{S}$ so $[\langle a' \; r' \rangle \;\; \langle a' \; b' \rangle] \in \sigma$. Therefore
$[\langle a' \; b' \rangle \;\; \langle a' \; r' \rangle] \in \sigma^{-1}$, and
$[\langle a \; b \rangle \;\; \langle a' \; b' \rangle] \in \bar{x}^C$ so
$[\langle a \; b \rangle \;\; \langle a' \; r' \rangle] \in \bar{x}^C \sigma^{-1}$. Hence
$[\langle a \; b \rangle \;\; \langle a' \; r' \rangle] \in \sigma^{-1} \bar{x}^K$ implies
$[\langle a \; b \rangle \;\; \langle a' \; r' \rangle] \in \bar{x}^C \sigma^{-1}$, and $x \in X_K$ is arbitrary so
$(\forall x)(x \in X_K \implies \sigma^{-1} \bar{x}^K \subseteq \bar{x}^C \sigma^{-1})$.

Considering now the nature of the relation σ from $S_A \times S_R$ to $S_A \times S_B$, clearly $D[\sigma] \subseteq S_A \times S_R$ so assume $\langle a \; r \rangle \in S_A \times S_R$. Since $R \leqslant^{\mathcal{S}} B$ the relation $\mathcal{S}$ from S_R to S_B has S_R as domain, so there is at least one $b \in S_B$ where $\langle r \; b \rangle \in \mathcal{S}$. Then $[\langle a \; r \rangle \;\; \langle a \; b \rangle] \in \sigma$ so $\langle a \; r \rangle \in D[\sigma]$, showing $S_A \times S_R \subseteq D[\sigma]$, so $D[\sigma] = S_A \times S_R$.

This confirms σ to be a weak homomorphism of $K = A \circ R$ to $C = A \circ B$, furthermore σ is one-many and this can be confirmed by assuming
$[\langle a \; b \rangle \;\; \langle a \; r \rangle] \in \sigma^{-1}$ and $[\langle a \; b \rangle \;\; \langle a^* \; r^* \rangle] \in \sigma^{-1}$.
Then $[\langle a \; r \rangle \;\; \langle a \; b \rangle] \in \sigma$, so $\langle r \; b \rangle \in \mathcal{S}$, furthermore
$[\langle a^* \; r^* \rangle \;\; \langle a \; b \rangle] \in \sigma$, in which case $a^* = a$ and
$\langle r^* \; b \rangle \in \mathcal{S}$. Hence $\langle b \; r \rangle \in \mathcal{S}^{-1}$ and
$\langle b \; r^* \rangle \in \mathcal{S}^{-1}$, however $\mathcal{S}$ is one-many so $\mathcal{S}^{-1}$ is a mapping, giving $r = r^*$. Therefore
$[\langle a \; b \rangle \;\; \langle a \; r \rangle] \in \sigma^{-1}$ and $[\langle a \; b \rangle \;\; \langle a^* \; r^* \rangle] \in \sigma^{-1}$
implies $\langle a \; r \rangle = \langle a^* \; r^* \rangle$, so σ^{-1} is a mapping.
Therefore σ is one-many, furthermore σ is a weak homomorphism of $K = A \circ R$ to $C = A \circ B$ so $A \circ R \leqslant^{\sigma} A \circ B$.

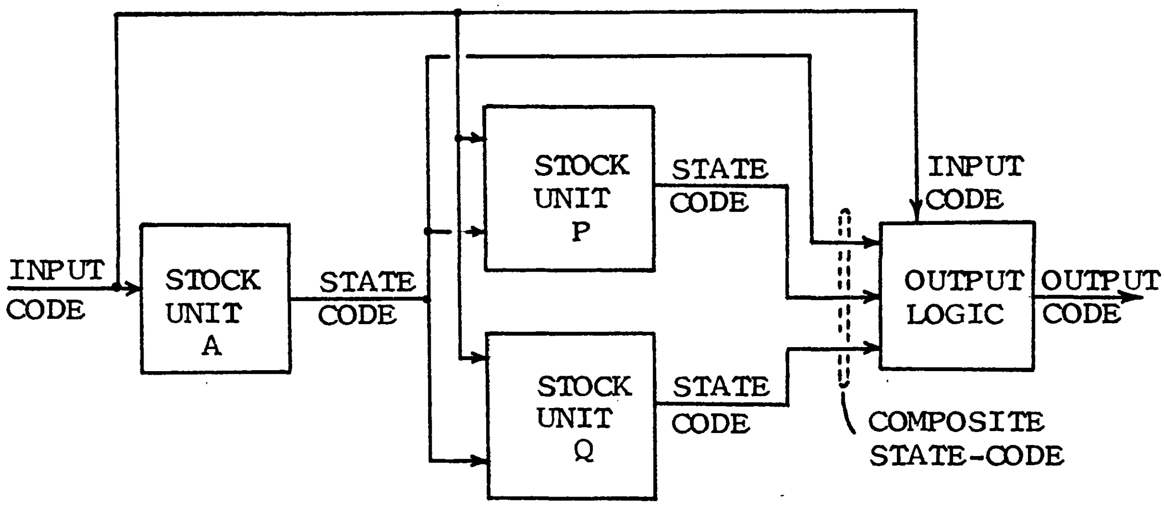
In conclusion $J \leqslant^{\omega} A \circ R$ and $A \circ R \leqslant^{\sigma} A \circ B$, that is $J \leqslant^{\omega} A \circ R \leqslant^{\sigma} A \circ B$, and covering is transitive so $J \leqslant^{\omega\sigma} A \circ B$. That is, the relation $\omega\sigma$ from S_J to $S_A \times S_B$ is a one-many weak homomorphism of J to $A \circ B$, and this completes the proof.

6.5 Complex Realisations

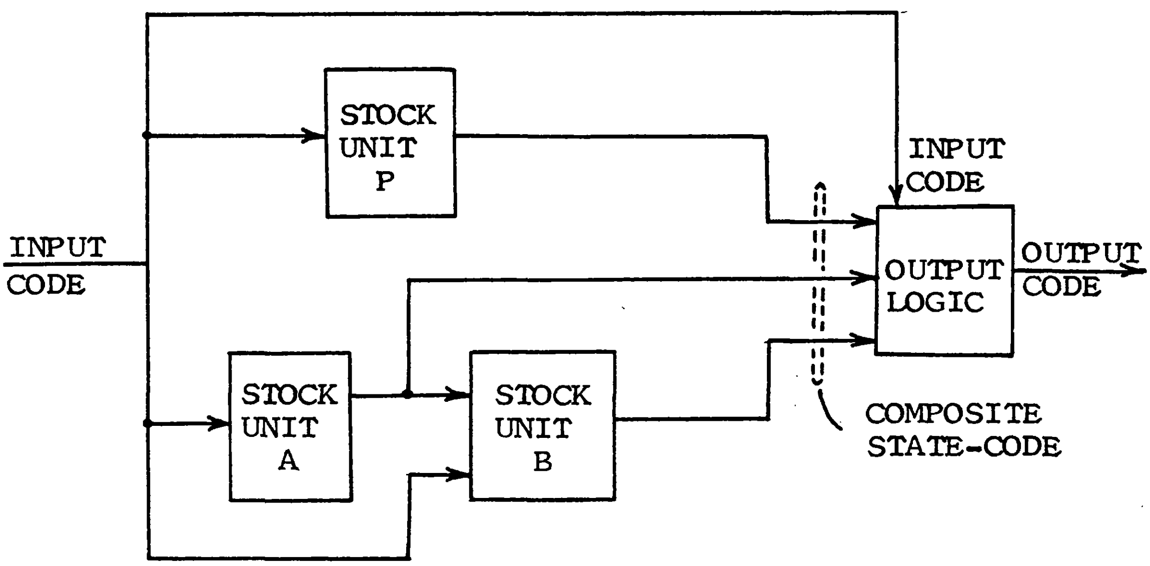
The preceeding establishes a systematic approach to automaton realisation using a cascade interconnection of stock units. Assuming $\hat{J}$ to be the objective automaton, the first step is to find a stock semiautomaton A covering some image F of the objective semiautomaton J. Then a semiautomaton R can be formalised so that $J \leqslant A \circ R$, and it then remains to realise R, by finding a stock semiautomaton B where $R \leqslant B$. Then $J \leqslant A \circ R \leqslant A \circ B$, giving $J \leqslant A \circ B$, so the stock units can be used in forming a cascade realisation of the objective automaton $\hat{J}$. Alternatively, it might be possible to use stock units to give a product realisation of $\hat{J}$. It has been seen that the approach then is to find a family of mutually-resolving image-semiautomata of the objective semiautomaton J, where each of these images is covered by some stock unit.

These studies establish direct and cascade realisations as separate approaches, however they are easily combined, for example consider the cascade realisation of a given automaton $\hat{J}$. Having determined a stock semiautomaton A covering an image F of the objective semiautomaton J, and having established a semiautomaton R so that $J \leqslant A \circ R$, it may be that no single stock semiautomaton covers R. However it may be that there are stock semiautomata P and Q where $R \leqslant P \times Q$, in which case $J \leqslant A \circ R \leqslant A \circ (P \times Q)$ so $J \leqslant A \circ (P \times Q)$, giving a "complex" cascade realisation as shown in figure 6.27(a). Alternatively, let F_u and F_v be mutually-resolving images of semiautomaton J, so that a product realisation can be formed by finding stock semiautomata P and Q where P covers F_u and Q covers F_v. Suppose however that P covers F_u, and that A and B are stock semiautomata where $F_v \leqslant A \circ B$. Then $J \leqslant P \times (A \circ B)$, and the stock semiautomata P, A and B can be used to give a complex product realisation as in figure 6.27(b).

Clearly product-realisation and cascade-realisation theory can be freely combined, to give realisations in the form of complex but "feedback free" interconnections of stock units. Furthermore, the designer need make no initial commitment as to the primary form of the realisation. The practical approach is to compare each of the available stock units against the images of the objective semiautomaton, as considered in section 6.3, and to list the covering stock units against each of the images. Attention can then be directed to the images themselves, to find a family of mutually-resolving images where each of these images is covered by some stock semiautomaton. This would give a product realisation of the objective automaton, but if no such family can be found a single image semiautomaton F,



(a) Realisation based on $J \leqslant A \circ (P \times Q)$



(b) Realisation based on $J \leqslant P \times (A \circ B)$

Figure 6.27

covered by some stock semiautomaton A, can be considered as the basis of a cascade or a "complex cascade" realisation.

Hence the design can proceed according to the suitability of the available stock units, that is according to the way these stock units relate to the images of the objective semiautomaton. This approach raises a particularly interesting problem concerning non-resolving images. Suppose it is found that a stock semiautomaton P covers a π_u-image F_u of the objective semiautomaton and that a stock semiautomaton Q covers a π_v-image F_v, but that the images are not fully resolving, that is $\pi_u * \pi_v = \pi_w$ where $\pi_w \neq 0(S_J)$. This means that the stock semiautomata P and Q cannot be used to form a product realisation of the objective automaton, since the composite state-codes do not all represent singleton ambiguities. However it might be possible to resolve the remaining ambiguities in a cascade, as shown in figure 6.28. Specifically, since $\pi_w = \pi_u * \pi_v$, it might be possible to form a π_w-image F_w so that $F_w \leqslant F_u \times F_v$. Then $F_u \leqslant P$ and $F_v \leqslant Q$ gives $F_u \times F_v \leqslant P \times Q$, consequently $F_w \leqslant F_u \times F_v \leqslant P \times Q$ so $F_w \leqslant P \times Q$, in which case a semiautomaton R can be formalised so that $J \leqslant (P \times Q) \circ R$. It then remains to realise semiautomaton R, by finding a stock semiautomaton B where $R \leqslant B$, and then $J \leqslant (P \times Q) \circ B$ so the realisation as in figure 6.28 can be formed.

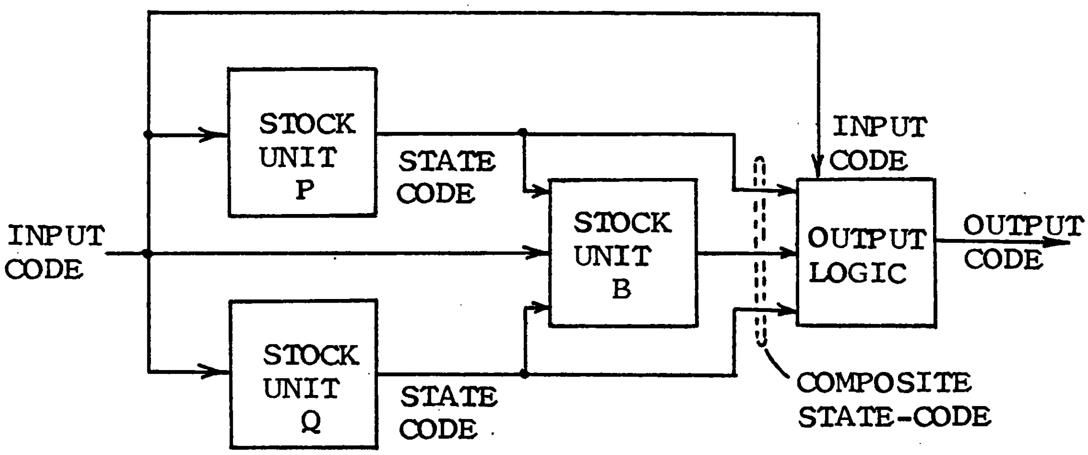


Figure 6.28   Realisation based on $J \leqslant (P \times Q) \circ B$

However a π_u-image $F_u = \langle S_u\ \overline{X}_u \rangle$ and a π_v-image $F_v = \langle S_v\ \overline{X}_v \rangle$, where $\pi_u * \pi_v = \pi_w$ and $\pi_w \neq 0(S_J)$, will not always produce such a realisation, since it will not always be possible to form a π_w-image F_w where $F_w \leqslant F_u \times F_v$. This depends on the relationship between $\pi_u \times \pi_v$ and $\pi_w = \pi_u * \pi_v$, where by definition

$$\pi_u * \pi_v = \left\{ f \;\middle|\; (\exists f_u)(\exists f_v) \begin{pmatrix} f_u \in \pi_u, f_v \in \pi_v, f = f_u \cap f_v \\ \& \ f \neq \emptyset \end{pmatrix} \right\}$$

Clearly π_w is closely related to $\pi_u \times \pi_v$, since any block $f_w \in \pi_w$ must have been formed by intersecting some block from π_u with some block from π_v, and this correspondence can be expressed as a relation ν from π_w to $\pi_u \times \pi_v$ where

$$\nu = \left\{ [f_w \ \ \langle f_u\ f_v \rangle] \;\middle|\; \begin{matrix} f_w \in \pi_w, \langle f_u\ f_v \rangle \in \pi_u \times \pi_v \\ \& \quad f_w = f_u \cap f_v \end{matrix} \right\}$$

Relation ν has domain $D[\nu] = \pi_w$, since $f_w \in \pi_w$ implies $f_w = f_u \cap f_v$ for some pair $\langle f_u\ f_v \rangle \in \pi_u \times \pi_v$, and has codomain $C[\nu] \subseteq \pi_u \times \pi_v$ since a pair $\langle f_u\ f_v \rangle \in \pi_u \times \pi_v$ might have a void intersection $f_u \cap f_v = \emptyset$. Furthermore the relation ν will usually be one-many, that is ν^{-1} will be a mapping, since for distinct pairs $\langle f_u\ f_v \rangle$, $\langle \mathfrak{f}_u\ \mathfrak{f}_v \rangle \in \pi_u \times \pi_v$ perhaps $f_u \cap f_v = f_w = \mathfrak{f}_u \cap \mathfrak{f}_v$ where $f_w \neq \emptyset$, in which case $[f_w\ \langle f_u\ f_v \rangle] \in \nu$ and $[f_w\ \langle \mathfrak{f}_u\ \mathfrak{f}_v \rangle] \in \nu$.

Suppose however that π_u and π_v are such that distinct pairs $\langle f_u\ f_v \rangle$, $\langle \mathfrak{f}_u\ \mathfrak{f}_v \rangle \in \pi_u \times \pi_v$ will always produce either void or distinct intersections, in which case relation ν from $\pi_w = \pi_u * \pi_v$ to $\pi_u \times \pi_v$ will be one-one rather than one-many. Then this will ensure that a π_w-image F_w of J can be formed so that $F_w \leqslant F_u \times F_v$, by defining $F_w = \langle S_w\ \overline{X}_w \rangle$ where $S_w = \pi_w$ and $X_w = X_J$, and defining the mappings $\overline{x}^w$ over π_w in accordance with the relation ν and the images F_u and F_v.

To see how these mappings are formed assume $x \in X_J$, and consider the assignment of a $\overline{x}^w$- successor to an arbitrary block f_w from the S_J-cover π_w. If $(f_w)\overline{x}^J = \emptyset$ no $\overline{x}^w$-successor is associated with the block $f_w \in \pi_w$, that is $f_w \notin D[\overline{x}^w]$, so assume instead $(f_w)\overline{x}^J \neq \emptyset$. Since $D[\nu] = \pi_w$ where ν is one-one there must be a particular pair $\langle f_u\ f_v \rangle \in \pi_u \times \pi_v$ where

$[f_w \ \langle f_u \ f_v \rangle] \in \nu$, in which case $f_w = f_u \cap f_v$ so $(f_w)\bar{x}^J = (f_u)\bar{x}^J \cap (f_v)\bar{x}^J$, and $(f_w)\bar{x}^J \neq \emptyset$ so $(f_u)\bar{x}^J \neq \emptyset$ and $(f_v)\bar{x}^J \neq \emptyset$. Furthermore $F_u = \langle S_u \ \bar{X}_u \rangle$ is a π_u-image of J, so $(f_u)\bar{x}^J \neq \emptyset$ implies $\langle f_u \ f'_u \rangle \in \bar{x}^u$ for some $f'_u \in \pi_u$ where $(f_u)\bar{x}^J \subseteq f'_u$, and similarly $F_v = \langle S_v \ \bar{X}_v \rangle$ is a π_v-image so $\langle f_v \ f'_v \rangle \in \bar{x}^v$ for some $f'_v \in \pi_v$ where $(f_v)\bar{x}^J \subseteq f'_v$. Then $[\langle f_u \ f_v \rangle \ \ \langle f'_u \ f'_v \rangle] \in \bar{x}^D$ as shown in figure 6.29, where $D = F_u \times F_v = \langle S_D \ \bar{X}_D \rangle$, in addition

$$(f_w)\bar{x}^J = (f_u)\bar{x}^J \cap (f_v)\bar{x}^J \subseteq f'_u \cap f'_v$$

so $(f_w)\bar{x}^J \subseteq f'_u \cap f'_v$.

$$\begin{tikzcd}
f_w \arrow[r, "\nu"] \arrow[d, "\bar{x}^w"'] & \langle f_u \ f_v \rangle \arrow[d, shift right, "\bar{x}^u"'] \arrow[d, shift left, "\bar{x}^v"] \arrow[d, bend left=60, "\bar{x}^D"] \\
f'_w & \langle f'_u \ f'_v \rangle \arrow[l, "\nu^{-1}"']
\end{tikzcd}$$

Figure 6.29

Furthermore $(f_w)\bar{x}^J \neq \emptyset$, so $f'_u \cap f'_v \neq \emptyset$, and this means that $f'_u \cap f'_v$ must be a block of π_w, that is $f'_u \cap f'_v = f'_w$ where f'_w is a block of π_w, so $[f'_w \ \langle f'_u \ f'_v \rangle] \in \nu$. Consequently $(f_w)\bar{x}^J \subseteq f'_u \cap f'_v$ becomes $(f_w)\bar{x}^J \subseteq f'_w$, and f'_w is taken to be the $\bar{x}^w$-successor of f_w, that is set $\langle f_w \ f'_w \rangle \in \bar{x}^w$. In summary, for any $x \in X_J$ and any

$f_w \in \pi_w$ where $(f_w)\bar{x}^J \neq \emptyset$, the $\bar{x}^w$-successor for f_w is determined by finding the particular pair $\langle f_u\ f_v \rangle \in \pi_u \times \pi_v$ where $[f_w\ \langle f_u\ f_v \rangle] \in \nu$, finding the pair $\langle f'_u\ f'_v \rangle \in \bar{x}^D$ where $[\langle f_u\ f_v \rangle\ \langle f'_u\ f'_v \rangle] \in \bar{x}^D$, finding the block f'_w from cover π_w where $[f'_w\ \langle f'_u\ f'_v \rangle] \in \nu$ and setting $\langle f_w\ f'_w \rangle \in \bar{x}^w$. Clearly $\bar{x}^w \subseteq \nu\, \bar{x}^D\, \nu^{-1}$, and the procedure gives the mapping $\bar{x}^w$ over π_w defined by

$$\bar{x}^w = \left\{ \langle f_w\ f'_w \rangle \;\middle|\; \begin{array}{c} \langle f_w\ f'_w \rangle \in \pi_w \times \pi_w,\ (f_w)\bar{x}^J \neq \emptyset \\ \&\quad \langle f_w\ f'_w \rangle \in \nu\bar{x}^D\nu^{-1} \end{array} \right\}$$

Then the semiautomaton $F_w = \langle S_w\ \overline{X}_w \rangle$, where $S_w = \pi_w$, $X_w = X_J$ and $\overline{X}_w$ is the set of the mappings $\bar{x}^w$ over π_w as defined above, is a π_w-image of the semiautomaton J. This can be confirmed by assuming $x \in X_J$, and assuming $f_w \in \pi_w$ where $(f_w)\bar{x}^J \neq \emptyset$. Then a $\bar{x}^w$-successor f'_w will have been associated with f_w, that is $\langle f_w\ f'_w \rangle \in \bar{x}^w$, and from above $(f_w)\bar{x}^J \subseteq f'_w$, confirming that F_w is an image semiautomaton of J. Consequently, the canonical relation Π_w from S_J to π_w will be a weak homomorphism of J to F_w. Moreover the relation ν will be a weak homomorphism of F_w to $F_u \times F_v$, and this can be demonstrated by assuming $x \in X_w$ and assuming $[\langle f_u\ f_v \rangle\ f'_w] \in \nu^{-1}\bar{x}^w$. Then $[\langle f_u\ f_v \rangle\ f_w] \in \nu^{-1}$, that is $[f_w\ \langle f_u\ f_v \rangle] \in \nu$, and $\langle f_w\ f'_w \rangle \in \bar{x}^w$, for some f_w, furthermore $\langle f_w\ f'_w \rangle \in \bar{x}^w$ implies $(f_w)\bar{x}^J \neq \emptyset$ and $\langle f_w\ f'_w \rangle \in \nu\,\bar{x}^D\,\nu^{-1}$. Consequently $[f_w\ \langle f_u\ f_v \rangle^*] \in \nu$, $[\langle f_u\ f_v \rangle^*\ \langle f'_u f'_v \rangle] \in \bar{x}^D$ and

$[\langle f'_u f'_v\rangle\ f'_w] \in \nu^{-1}$ for some $\langle f_u\ f_v\rangle^*$ and some $\langle f'_u f'_v\rangle$, however ν is one-one so $[f_w\ \langle f_u\ f_v\rangle\ ] \in \nu$ and $[f_w\ \langle f_u\ f_v\rangle^*] \in \nu$ implies $\langle f_u\ f_v\rangle = \langle f_u\ f_v\rangle^*$. Therefore $[\langle f_u\ f_v\rangle\ \ \langle f'_u f'_v\rangle] \in \overline{x}^D$, and $[\langle f'_u f'_v\rangle\ f'_w] \in \nu^{-1}$ so $[\langle f_u\ f_v\rangle\ \ f'_w] \in \overline{x}^D\ \nu^{-1}$. Hence $[\langle f_u\ f_v\rangle\ \ f'_w] \in \nu^{-1}\overline{x}^w$ implies $[\langle f_u\ f_v\rangle\ \ f'_w] \in \overline{x}^D\ \nu^{-1}$, and $x \in X_w$ is arbitrary so $(\forall x)(x \in X_J \implies \nu^{-1}\overline{x}^w \subseteq \overline{x}^D\ \nu^{-1})$. Then since $D[\nu] = \pi_w$ the relation ν is a weak homomorphism of F_w to $D = F_u \times F_v$, indeed ν is one-one (but is not surjective) and is therefore a partial monomorphism of F_w to $F_u \times F_v$.

Consequently $F_w \leqslant F_u \times F_v$, and assuming P and Q are stock semiautomata where $F_u \leqslant P$ and $F_v \leqslant Q$ then $F_w \leqslant F_u \times F_v \leqslant P \times Q$, giving $F_w \leqslant P \times Q$. This means that a semiautomaton R can be formalised so that $J \leqslant (P \times Q) \circ R$, and finding a stock semiautomaton B where $R \leqslant B$ will give a realisation $J \leqslant (P \times Q) \circ B$. In fact the relation ν from π_w to $\pi_u \times \pi_v$ need not be one-one. The relation will usually be one-many, and this will still give a covering $F_w \leqslant F_u \times F_v$, so long as ν relates just one pair $\langle f_u\ f_v\rangle \in \pi_u \times \pi_v$ with any block f_w from π_w where $(f_w)\overline{x}^J \neq \emptyset$. That is, if $f_w \in \pi_w$ where $(f_w)\overline{x}^J \neq \emptyset$, there must be just one pair $\langle f_u\ f_v\rangle$ where $[f_w\ \ \langle f_u\ f_v\rangle\ ] \in \nu$.

Theorem

Let $F_u = \langle S_u\ \overline{X}_u \rangle$ be a π_u-image of a semiautomaton $J = \langle S_J\ \overline{X}_J \rangle$, and let $F_v = \langle S_v\ \overline{X}_v \rangle$ be a π_v-image of J. Define $\pi_w = \pi_u * \pi_v$, and assume π_w is such that for any $f_w \in \pi_w$ and any $x \in X_J$, $(f_w)\overline{x}^J \neq \emptyset$ implies $f_w = f_u \cap f_v$ for just one pair $\langle f_u\ f_v \rangle \in \pi_u \times \pi_v$.

Then there exists a π_w-image $F_w = \langle S_w\ \overline{X}_w \rangle$, such that $F_w \leqslant F_u \times F_v$.

Proof

Define $D = \langle S_D\ \overline{X}_D \rangle$ where $D = F_u \times F_v$, so $S_D = \pi_u \times \pi_v$, $X_D = X_J$, and $x \in X_J$ implies $\overline{x}^D \in \overline{X}_D$ where

$$\overline{x}^D = \left\{ [\langle f_u\ f_v \rangle\ \ \langle f'_u\ f'_v \rangle] \middle| \langle f_u\ f'_u \rangle \in \overline{x}^u \ \&\ \langle f_v\ f'_v \rangle \in \overline{x}^v \right\}.$$

Define the relation ν from π_w to $\pi_u \times \pi_v$ where

$$\nu = \left\{ [f_w\ \ \langle f_u\ f_v \rangle\ ] \middle| \begin{array}{l} f_w \in \pi_w,\ \langle f_u\ f_v \rangle \in \pi_u \times \pi_v, \\ \&\ \ f_w = f_u \cap f_v \end{array} \right\}$$

Assuming $[\langle f_u\ f_v \rangle\ \ f_w] \in \nu^{-1}$ and $[\langle f_u\ f_v \rangle\ \ f_w^*] \in \nu^{-1}$ then $f_w = f_u \cap f_v = f_w^*$, so $f_w = f_w^*$, and this confirms ν^{-1} to be a mapping so relation ν is one-many. Clearly the domain $D[\nu]$ of ν is a subset of π_w, furthermore $\pi_w = \pi_u * \pi_v$ so $f_w \in \pi_w$ implies $f_w = f_u \cap f_v$ for some $f_u \in \pi_u$ and some $f_v \in \pi_v$, in which case $[f_w\ \ \langle f_u\ f_v \rangle\ ] \in \nu$. Then $f_w \in D[\nu]$ so $\pi_w \subsetneq D[\nu]$, and this confirms $D[\nu] = \pi_w$.

Define $F_w = \langle S_w\ \overline{X}_w \rangle$ where $S_w = \pi_w$ and $X_w = X_J$, so $x \in X_J$ implies $\overline{x}^w \in \overline{X}_w$, and let $\overline{x}^w$ be the mapping over π_w defined by

$$\bar{x}^w = \left\{ \langle f_w\ f'_w \rangle \;\middle|\; \begin{array}{l} \langle f_w\ f'_w \rangle \in \Pi_w \times \Pi_w,\ (f_w)\bar{x}^J \neq \emptyset \\ \quad \& \quad \langle f_w\ f'_w \rangle \in \nu \bar{x}^D \nu^{-1} \end{array} \right\}$$

Clearly $(f_w)\bar{x}^J = \emptyset$ implies $f_w \notin D[\bar{x}^w]$, so to confirm F_w to be a Π_w-image assume $x \in X_J$, $f_w \in \Pi_w$ and $(f_w)\bar{x}^J \neq \emptyset$. Then there must exist a particular pair $\langle f_u\ f_v \rangle \in \Pi_u \times \Pi_v$ where $f_w = f_u \cap f_v$, in which case $[f_w\ \langle f_u\ f_v \rangle] \in \nu$. Furthermore $f_w = f_u \cap f_v$ implies $(f_w)\bar{x}^J = (f_u)\bar{x}^J \cap (f_v)\bar{x}^J$, and $(f_w)\bar{x}^J \neq \emptyset$ so $(f_u)\bar{x}^J \neq \emptyset$ and $(f_v)\bar{x}^J \neq \emptyset$. Since F_u is an image of semiautomaton J, $(f_u)\bar{x}^J \neq \emptyset$ implies $\langle f_u\ f'_u \rangle \in \bar{x}^u$ for some f'_u where $(f_u)\bar{x}^J \subseteq f'_u$, similarly F_v is an image of J so $\langle f_v\ f'_v \rangle \in \bar{x}^v$ for some f'_v where $(f_v)\bar{x}^J \subseteq f'_v$. Therefore $(f_w)\bar{x}^J = (f_u)\bar{x}^J \cap (f_v)\bar{x}^J \subseteq f'_u \cap f'_v$, furthermore $(f_w)\bar{x}^J \neq \emptyset$ so $f'_u \cap f'_v \neq \emptyset$, and $\Pi_w = \Pi_u * \Pi_v$ so $f'_u \cap f'_v = f'_w$ for some $f'_w \in \Pi_w$, in which case $(f_w)\bar{x}^J \subseteq f'_w$ and $[f'_w\ \langle f'_u\ f'_v \rangle] \in \nu$.

Consequently $[f_w\ \langle f_u\ f_v \rangle] \in \nu$, $[\langle f_u\ f_v \rangle\ \langle f'_u\ f'_v \rangle] \in \bar{x}^D$ and $[\langle f'_u\ f'_v \rangle\ f'_w] \in \nu^{-1}$ in which case $\langle f_w\ f'_w \rangle \in \nu\bar{x}^D\nu^{-1}$, and then $\langle f_w\ f'_w \rangle \in \bar{x}^w$ since $(f_w)\bar{x}^J \neq \emptyset$. Hence $(f_w)\bar{x}^J \neq \emptyset$ implies $\langle f_w\ f'_w \rangle \in \bar{x}^w$ for some f'_w where $(f_w)\bar{x}^J \subseteq f'_w$, and this confirms F_w to be a Π_w-image of semiautomaton J.

To show that ν is a weak homomorphism of F_w to $F_u \times F_v$, assume $x \in X_J$ and assume $[\langle f_u\ f_v \rangle\ f'_w] \in \nu^{-1}\bar{x}^w$. Then $[\langle f_u\ f_v \rangle\ f_w] \in \nu^{-1}$ and $\langle f_w\ f'_w \rangle \in \bar{x}^w$ for some f_w, furthermore $\langle f_w\ f'_w \rangle \in \bar{x}^w$ implies $(f_w)\bar{x}^J \neq \emptyset$ and $\langle f_w\ f'_w \rangle \in \nu\bar{x}^D\nu^{-1}$. Then $[f_w\ \langle f_u\ f_v \rangle^*] \in \nu$,

$[\langle f_u\ f_v\rangle^* \ \ \langle f'_u\ f'_v\rangle] \in \bar{x}^D$ and $[\langle f'_u\ f'_v\rangle\ f'_w] \in \nu^{-1}$,
for some $\langle f_u\ f_v\rangle^*$ and some $\langle f'_u\ f'_v\rangle$, however
$(f_w)\bar{x}^J \neq \emptyset$ implies $f_w = f_u \cap f_v$ for a particular pair
$\langle f_u\ f_v\rangle \in \pi_u \times \pi_v$. That is, if $(f_w)\bar{x}^J \neq \emptyset$ where
$[f_w\ \langle f_u\ f_v\rangle] \in \nu$ and $[f_w\ \langle f_u\ f_v\rangle^*] \in \nu$, then
$\langle f_u\ f_v\rangle' = \langle f_u\ f_v\rangle^*$. Hence
$[\langle f_u\ f_v\rangle^* \ \ \langle f'_u\ f'_v\rangle] \in \bar{x}^D$ becomes
$[\langle f_u\ f_v\rangle \ \ \langle f'_u\ f'_v\rangle] \in \bar{x}^D$, and $[\langle f'_u\ f'_v\rangle\ f'_w] \in \nu^{-1}$ so
$[\langle f_u\ f_v\rangle\ f'_w] \in \bar{x}^D\ \nu^{-1}$.

Therefore $[\langle f_u\ f_v\rangle\ f'_w] \in \nu^{-1}\bar{x}^w$ implies
$[\langle f_u\ f_v\rangle\ f'_w] \in \bar{x}^D\ \nu^{-1}$, and $x \in X_J$ is arbitrary so
$(\forall x)(x \in X_J \implies \nu^{-1}\bar{x}^w \subseteq \bar{x}^D\ \nu^{-1})$. In addition
$D[\nu] = \pi_w$, and this confirms ν to be a weak
homomorphism of F_w to $F_u \times F_v$, furthermore ν is one-many
so $F'_w \leqslant^{\nu} F_u \times F_v$, completing the proof.

6.6 Conclusion

The "algebraic structure theory" of finite automata, initiated by J. Hartmanis, provides an approach to automaton decomposition [Hartmanis (a)] and to state-assignment [Hartmanis (b); Stearns & Hartmanis]. The aim, in decomposing a given automaton, is to form an equivalent interconnection of smaller automata, and such a decomposition can usually be formed using preserved partitions. However this is not the most general approach, since decompositions can also be based on preserved covers [Hennie]. Further problems arise when the given automaton is partial, since then the normal symbolism becomes inadequate, and the meaning of the "realisation" concept must be reconsidered.

These problems require a more detailed analysis, and this has been given by Yoeli in a definitive study of cascade decomposition [Yoeli]. The aim, in the present chapter, has been to adopt this approach in forming composite realisations using units from stock. The chapter can be regarded as a generalisation of loop-free structure theory [Hartmanis (a)] since cascade, product, and more complex realisations have been considered, however there is a fundamental distinction between automaton decomposition and the present composite-realisation approach. A composite realisation using stock units must be formed with regard to the available components, and the approach requires the properties of the objective automaton to be related to the stock semiautomata. In automaton decomposition, however, attention is restricted to the objective automaton, so decomposition can be regarded as automaton analysis, rather than the establishment of relationships between semiautomata. An additional distinction arises if the stock units are expressed as semiautomata, rather than automata. Constructions using automata [Booth; Hartmanis & Stearns] can then be disregarded, and attention can be restricted to "composite semiautomata".

The problem considered in the present chapter is of increasing practical importance, since there is an increasing need to relate MSI units from stock to the design objective, and to establish a suitable interconnection. The essential steps in this approach have been developed formally, since formal approaches to circuit design present the real intellectual challenge, in addition a formal approach can give solutions where none is intuitively obvious, and might easily be translated into a program for a digital computer. A formal approach has several disadvantages, however, for example it has been seen that each stock unit might have several modes of operation, as directed by "auxiliary inputs", and in a formal analysis each mode must be considered separately. Furthermore the "auxiliary outputs" have been disregarded, whereas i~~s~~n practice these are particularly important, for example the "carry" output from a counter can be connected to the "load" input to give a count of smaller modulus. However a more serious problem relates to the complexity of modern digital networks, since a typical sequential circuit might be a mixture of synchronous and asynchronous systems, with the synchronous units driven from several separate clocks. Consequently a formal approach cannot match the art of the experienced circuit designer, but should instead be regarded as augmenting the purely intuitive approach, possibly as the basis of an interactive computer program.

Composite realisation studies complete the present work, however it can be seen that these studies have been of prime influence throughout the preceeding chapters. Before composite realisations were considered the basic interconnection schemes for semiautomata were presented and the "product" and "cascade" constructions, using partial semiautomata, were formalised.

These constructions were essential to the final chapter, and the additional aim was to clarify the relationships between the composite and the component semiautomata. It was important also to clarify the meaning of a "realisation" of a given automaton, and the aim in the fourth chapter was to present a formal realisation approach. The direct comparison of an objective semiautomaton with available stock units was developed, and it was also shown that reduction of the objective automaton should be carefully considered, since the reduction will relate to the stock units in a different way.

Before automaton realisation can proceed, however, the design objective must be clearly specified. It has been shown that this design objective can be visualised as an objective translation, from input tapes to output symbols, however the translation cannot be directly converted into appropriate hardware. The objective translation must first be expressed in the form of a finite automaton, so that a one-many weak homomorphism can be used to convert this "objective automaton" into a hardware realisation. The true meaning of the objective automaton is not always evident, however, and there is particular confusion regarding the meaning of the "states".

In considering this problem the "states" were replaced by state-events, and the objective automaton was regarded as an expression of a right-invariant equivalence, in accordance with the Nerode theorem. The equivalence classes are then of particular interest since they must be "regular", and can be defined as regular expressions. Consequently the objective translation can be formalised using regular expressions, with one regular expression defining each of the objective output events, and the regular expressions can be used to form an appropriate "event automaton". The conversion from regular expressions to this multiple-output objective automaton will often be laborious, nevertheless it was thought important to consider the procedure, with the added complication of "invalid" tapes, in some detail, since this seems the only alternative to the purely intuitive approach.

The opening chapters are less directly related to the circuit design problem, the aim being the introduction of convenient symbology, and the introduction of fundamental concepts on which a study of circuit design can be based. A particularly important feature is the formalisation of state-transition systems as indexed unary algebras, for example the unary algebra $\langle S_A \overline{X}_A \rangle$ is a X_A-semiautomaton over S_A, meaning that each input symbol from X_A indexes a mapping over the state-set S_A. Consequently the study of the state-transition aspect of an automaton can be based on universal algebra, so that attention can be restricted to fundamental issues, and semiautomaton properties can be related to properties of other algebras. More significantly an approach based on universal algebra encourages distinction between the properties of a finite automaton and those associated with the semi-automaton "core", for example the Nerode theorem relates to a fundamental property of finite automata, whereas the automorphism group of an automaton [Fleck] relates to the semi-automaton core, and should not be regarded as particular to finite automaton theory.

Considered in the broader context, as a contribution to sequential-circuit studies, the work of Hartmanis and Stearns has initiated an engineering interest in the abstract. The structure theory introduces abstract concepts of fundamental importance, for example it becomes important to visualise homomorphic images, and to visualise congruences, partial orderings and lattices. The same abstract ideas are of importance throughout discrete science, for example in the theory of computation, and the same formal mathematical approach pervades much of the research of current interest [Arbib; Brzozowski & Yoeli; Krohn & Rhodes; Ginsburg].

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APPENDIX A Set Theory [Halmos; Suppes (a)]

As in normal logical symbolism, the symbol $\Longleftrightarrow$ denotes logical equivalence or "if and only if", often abbreviated as "iff". Similarly, the symbol $\Longrightarrow$ denotes implication. The universal quantifier is shown as $\forall$ and the existential quantifier as $\exists$, for example $(\forall x)$ means "for all x" and $(\exists x)$ means "there exists some x". [Suppes (b)].

The "cardinality" of a set S, denoted $|S|$, is the number of elements within the set S. A set is a "finite" set iff it has finite cardinality, and is otherwise an "infinite" set. A set with unity cardinality is a "singleton".

For sets A and B, $A \subseteq B$ denotes that A is a "subset" of B, $A = B$ denotes that A and B are "identical" sets (otherwise A and B are "distinct", denoted $A \neq B$), and $A \subset B$ denotes that A is a "proper" subset of B, where

$$A = B \Longleftrightarrow (\forall x)\ (x \in A \Longleftrightarrow x \in B)$$

$$A \subseteq B \Longleftrightarrow (\forall x)\ (x \in A \Longrightarrow x \in B)$$

and

$$A \subset B \Longleftrightarrow A \subseteq B \ \ \&\ \ A \neq B$$

The "null set", denoted $\emptyset$, is the set having zero cardinality and is a subset of all sets. A set S is "nonvoid" iff $S \neq \emptyset$. The set $\mathfrak{P}(S) = \{P | P \subseteq S\}$ is the "power set" of S, for example $\emptyset \in \mathfrak{P}(S)$ and $S \in \mathfrak{P}(S)$.

For sets A and B, the set $A \cap B = \{x | x \in A \ \& \ x \in B\}$ is the "intersection" of A and B and these sets are "disjoint" iff $A \cap B = \emptyset$. The set $A \cup B = \{x | x \in A \text{ or } x \in B\}$ is the "union" of the sets A, B.

The "Cartesian product" associated with a pair $\langle X, Y\rangle$ of sets is the set $X x Y = \{\langle x, y\rangle | x \in X \ \& \ y \in Y\}$, and $X^n = \{\langle x_1, x_2, \ldots, x_n\rangle | x_1, x_2, \ldots, x_n \in X\}$ for any set X and any $n > o$.

A set ρ is a "relation" iff

$(\forall x) \ (x \in \rho \Longrightarrow (\exists y)(\exists z)(x = \langle y \ z\rangle))$.

Then the set $D[\rho] = \{y | (\exists z)(\langle y \ z\rangle \in \rho)\}$ is the "domain" of the relation ρ, and the set $C[\rho] = \{z | (\exists y)(\langle y \ z\rangle \in \rho)\}$ is the "codomain" of ρ. Relation ρ is a relation from Y to Z iff $D[\rho] \subseteq Y$ and $C[\rho] \subseteq Z$. This can be denoted $Y\rho Z$, and similarly $y\rho z$ can be used to denote $\langle y \ z\rangle \in \rho$. A relation ρ from Y to Z is "complete" from Y iff $D[\rho] = Y$, and is a relation "onto" Z iff $C[\rho] = Z$. A relation ρ from Y to Y is a relation "over" Y, and a relation from Y^n to Y is a "n-ary" relation over Y. Such a relation is "finitary" iff n if finite.

For Y' any set, the set

$(Y')\rho = \{z | (\exists y)(y \in Y' \ \& \ \langle y \ z\rangle \in \rho)\}$

is the "image" of the set Y under the relation ρ. Substituting $\{y\}$ for Y' gives $(\{y\})\rho = \{z | \langle y \ z\rangle \in \rho\}$ and for convenience this set can be denoted $[y]\rho$, so $[y]\rho = \{z | \langle y \ z\rangle \in \rho\}$ is the image under ρ of the singleton $\{y\}$.

For any relation ρ from Y to Z, the set

$\rho^{-1} = \{\langle z\ y\rangle \mid \langle y\ z\rangle \in \rho\}$ is the "converse" of ρ and is a relation from Z to Y.

For any relations ρ and μ, the set $\rho.\mu = \{\langle x\ z\rangle \mid (\exists y)(\langle x\ y\rangle \in \rho\ \&\ \langle y\ z\rangle \in \mu)\}$ is the "composition" of ρ and μ. For relations μ, ρ and ω, set theory verifies

$$(\mu^{-1})^{-1} = \mu,$$

$$(\mu.\rho)^{-1} = \rho^{-1}.\mu^{-1},$$

$$\mu.(\rho.\omega) = (\mu.\rho).\omega,$$

furthermore $\mu \subseteq \rho$ iff $\mu^{-1} \subseteq \rho^{-1}$.

For S any set, the set $\Delta[S] = \{\langle s\ s\rangle \mid s \in S\}$ is the "diagonal" over S and is a relation over S.

For any relation ρ, $\Delta[D[\rho]] \subseteq \rho.\rho^{-1}$ and $\Delta[C[\rho]] \subseteq \rho^{-1}.\rho$. For a relation ρ from Y to Z, $\Delta[Y].\rho = \Delta[D[\rho]].\rho = \rho = \rho.\Delta[C[\rho]] = \rho.\Delta[Z]$. For S any set and ρ any relation, the relation $\rho|S = \Delta[S].\rho$ is the "restriction" of ρ to S and is a subset of ρ. Equivalently, $\rho|S = \{\langle y\ z\rangle \mid \langle y\ z\rangle \in \rho\ \&\ y \in S\}$.

A relation ρ is a mapping iff $(\forall y)(\forall z_1)(\forall z_2)(y\rho z_1\ \&\ y\rho z_2 \Rightarrow z_1 = z_2)$, and the converse of a mapping is said to be "one-many". A mapping ρ from Y to Z can be denoted $\rho: Y \longrightarrow Z$. Such a mapping is a "surjection" or is "surjective" iff $C[\rho] = Y$, and is an "injection" or is "injective" iff the converse ρ^{-1} is also a mapping. A mapping is a "bijection" or is "bijective" iff both surjective and injective.

A mapping with domain I and codomain X can be called a "family", and the set I will be the "index set". Such a family can be denoted $\{x_i\}_I$, where x_i denotes the element of codomain X related by the mapping to the element $i \in I$. For $I = \{0, 1, \ldots n-1\}$, the family $\{x_i\}_I$ is given by

$$\{x_i\}_I = \{\langle 0\ x_0\rangle \langle 1\ x_1\rangle \ldots\ldots \langle n-1\ x_{n-1}\rangle\}$$ and can be represented as $\langle x_0\ x_1\ x_2\ \ldots\ldots\ x_{n-1}\rangle$.

If $\{S_i\}_I$ is a family of sets where $I = \{0,1,\ldots n-1\}$, the "union over the family" is the set $\cup\{S_i\}_I = S_0 \cup S_1 \cup \ldots \cup S_{n-1}$ and the "product over the family" is the set $\times\{S_i\}_I = S_0 \times S_1 \times \ldots \times S_{n-1}$.

APPENDIX B Universal Algebra [Birkhoff; Cohn; Gratzer]

Universal algebra is the study of fundamental features common to groups, semigroups, Boolean algebras, lattices, rings and other algebras. A "universal algebra" is a pair $\langle A\ F_A \rangle$, where A is a nonvoid set and F_A is a family of finitary mappings over A, the index set being F. If the family F_A has index set $F = \{f', f'' \ldots.\}$, the family can be shown as $\langle f'_A\ f''_A \ldots. \rangle$ where $f'_A, f''_A, \ldots.$ are finitary mappings over A. Alternatively, consider a family $\langle \bar{x}_1\ \bar{x}_2\ \bar{x}_3 \ldots.. \rangle$ of unary mappings over a set S, where the index set is $X = \{x_1, x_2, x_3, \ldots\}$. Then each $x \in X$ indexes a unary mapping $\bar{x}$, and the unary algebra can be expressed as $\langle S\ \bar{X} \rangle$, where $\bar{X}$ is the set of the indexed mappings.

The features of mappings (or "functions") are important in the study of any discrete system. For example, the kernel $\ker(f) = f.f^{-1}$ of a mapping f is an equivalence over the domain $D[f]$. Furthermore a composition of mappings is itself a mapping, and a composition of surjections of a set onto itself is again a surjection of the set onto itself. These surjections can be said to be "closed" under composition, furthermore composition is always associative, so the surjections of any set onto itself will form a "semigroup". Similarly the injections of a set S into itself form an "injection semigroup", and the bijections form a group, since $\Delta[S]$ is an identity bijection and the inverse of a bijection B is the bijection B^{-1}.

A relation over a given set is a "partial ordering" iff reflexive, antisymmetric and transitive, and a "lattice" is a set L together with a partial ordering, such that any two elements from L have a "least upper bound" and a "greatest lower bound". For example the inclusion relation is a partial ordering over $\mathfrak{P}(A)$, where $\mathfrak{P}(A)$ is the set of all the subjects of a set A, furthermore arbitrary subsets $x, y \in \mathfrak{P}(A)$ have $x \cup y$ as least upper bound and have $x \cap y$ as greatest lower bound, consequently $\langle \mathfrak{P}(A) \subseteq \rangle$ is a lattice. Similarly $\mathfrak{P}(A \times A)$ forms a lattice, that is the relations over the set A form a lattice, and the equivalence relations form a sublattice. A lattice L is "complete" if any subset $L' \subseteq L$ has a least upper bound and a greatest lower bound. Every finite lattice is complete, for example the equivalences over a given set form a complete lattice.

The above concepts are fundamental, and provide a basis for considering the three most important algebraic concepts, these being subalgebra, congruence and homomorphism. For an algebra $\langle A\ F_A \rangle$, an equivalence R over A is a "congruence" iff preserved under each mapping from F_A. That is, for an arbitrary n-ary mapping $f_A \in F_A$, if $\langle a_o \ldots a_{n-1} \rangle \in A^n$ and $\langle a'_o \ldots a'_{n-1} \rangle \in A^n$ where $a_o R a'_o$, $a_1 R a'_1, \ldots a_{n-1} R a'_{n-1}$, and if f_A assigns $\langle a_o \ldots a_{n-1} \rangle$ to $a \in A$ and assigns $\langle a'_o \ldots a'_{n-1} \rangle$ to $a' \in A$, then $a R a'$. For algebras $\langle A\ F_A \rangle$ and $\langle B\ F_B \rangle$, an arbitrary index term $f \in F$ indexes a mapping $f_A \in F_A$ and a mapping $f_B \in F_B$. In using the same index set for more than one algebra, it must be assumed that "commonly indexed" mappings have common arity. For example the above $f \in F$

indexes $f_A \in F_A$ and $f_B \in F_B$, so both these mappings must be n-ary for some n, giving $f_A : A^n \longrightarrow A$ and $f_B : B^n \longrightarrow B$. In particular, the algebra $\langle B\ F_B \rangle$ is a "subalgebra" of the algebra $\langle A\ F_A \rangle$ iff $B \subseteq A$ and, for any commonly-indexed n-ary mappings f_A and f_B, $f_B = f_A | B^n$. For a semiautomaton $A = \langle S_A\ \bar{X}_A \rangle$ and a semiautomaton $B = \langle S_B\ \bar{X}_B \rangle$, where $X_A = X_B = X$, B is a "subsemiautomaton" of A iff $S_B \subseteq S_A$ and, for any $x \in X$, $\bar{x}^B = \bar{x}^A | S_B$.

For an algebra $\langle A\ F_A \rangle$ and an algebra $\langle B\ F_B \rangle$, let H be a mapping of the set A into the set B. Then H defines a mapping H(n) of A^n to B^n so that H(n) assigns an element $\langle a_o \ldots a_{n-1} \rangle \in A^n$ to the specific element $\langle b_o \ldots b_{n-1} \rangle \in B^n$ where $\langle a_o\ b_o \rangle, \langle a_1\ b_1 \rangle, \ldots \langle a_{n-1}\ b_{n-1} \rangle \in H$. The mapping H is a "homomorphism" of the algebra $\langle A\ F_A \rangle$ to the algebra $\langle B\ F_B \rangle$ iff, for any commonly-indexed n-ary mappings $f_A : A^n \longrightarrow A$ and $f_B : B^n \longrightarrow B$, $f_A . H = H(n) . f_B$. In particular, a mapping $H : S_A \longrightarrow S_B$ is a homomorphism of a semiautomaton $\langle S_A\ \bar{X}_A \rangle$ to a semiautomaton $\langle S_B\ \bar{X}_B \rangle$, assuming $X_A = X_B = X$, iff $\bar{x}^A H = H \bar{x}^B$ for any $x \in X$. A surjective homomorphism is an "epimorphism", an injective homomorphism is a "monomorphism" and a bijective homomorphism is an "isomorphism". A homomorphism of an algebra to itself is an "endomorphism", and an isomorphism of an algebra to itself is an "automorphism".

A composition of homomorphisms produces a homomorphism, the kernel of a homomorphism is a congruence, and the image of a homomorphism is a subalgebra of the target. Furthermore every quotient algebra is a homomorphic image of the parent algebra, in particular the canonical surjection from a semiautomaton to a quotient semiautomaton is a homomorphism. The congruences of an algebra form a lattice, furthermore the preserved partitions of a semiautomaton are in one-one correspondence with the congruences, and are similarly ordered, so the preserved partitions form a lattice. The endomorphisms of any algebra $\langle A\ F_A \rangle$ form a semigroup, specifically a "monoid" since the diagonal $\Delta[A]$ is an identity, and similarly the automorphisms of any algebra form a group.

Reconsidering the properties of mappings, a mapping f of a set A to a set B has kernel $\ker(f) = f \cdot f^{-1}$ where $\ker(f)$ is an equivalence over A. Then there exists a decomposition $f = \rho f' \mu$ where ρ is the natural surjection of A onto the partition $A/\ker(f)$, f' is a bijection between $A/\ker(f)$ and the subset $(A)f \subseteq B$, and the "inclusion" mapping μ of $(A)f$ into B is an injection. Similarly, for a homomorphism H of an algebra $\langle A\ F_A \rangle$ to an algebra $\langle B\ F_B \rangle$, there exists a decomposition $H = \rho H' \mu$ where ρ is the natural epimorphism of A onto the quotient algebra $A/\ker(H)$, the mapping H' is an isomorphism between $A/\ker(H)$ and the subalgebra formed by the image of the homomorphism H, and μ is the inclusion injection of the subset $(A)H \subseteq B$ into B. If R and R' are congruences of an algebra $\langle A\ F_A \rangle$ where $R \subseteq R'$, there is a homomorphism H of the quotient algebra A/R to the quotient algebra A/R'. Furthermore, the composition of the epimorphism of A to A/R with the homomorphism H gives the epimorphism of A to A/R'.

An algebra $\langle A\ F_A \rangle$ is partial if the mappings forming F_A are not all complete, that is a n-ary mapping $f_A \in F_A$ might have domain $D[f_A] \neq A^n$. The above properties do not usually generalise to partial algebras, the study of partial algebras being particularly complex since various definitions can be given for the fundamental concepts, in particular for subalgebras, congruences and homomorphisms [Gratzer].

APPENDIX C Greek Alphabet

A, α	Alpha
B, β	Beta
Γ, γ	Gamma
Δ, δ	Delta
E, ϵ	Epsilon
Z, ζ	Zeta
H, η	Eta
Θ, θ	Theta
I, ι	Iota
K, κ	Kappa
Λ, λ	Lambda
M, μ	Mu
N, ν	Nu
Ξ, ξ	Xi
O, o	Omicron
Π, π	Pi
P, ρ	Rho
Σ, σ	Sigma
T, τ	Tau
Υ, υ	Upsilon
Φ, ϕ	Phi
X, χ	Chi
Ψ, ψ	Psi
Ω, ω	Omega